ISRAEL JOURNAL OF MATHEMATICS 110 (1999), 189–218

THE CENTRAL LIMIT PROBLEM ON LOCALLY COMPACT GROUPS

BY

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ABSTRACT

It is shown that the limit μ of a commutative infinitesimal triangular system Δ on a totally disconnected locally compact group G is embeddable in a continuous one-parameter convolution semigroup if either (1)G is a compact extension of a closed solvable normal subgroup or (2)G is discrete and Δ is normal or (3)G is a discrete linear group over a field of characteristic zero. For a special triangular system of convolution powers ($\mu_n^{k_n} \to \mu, \mu_n \to \delta_3$), the above is shown to hold without any of the conditions (1)–(3). For a general locally compact group G necessary conditions are obtained for the embeddability of a shift of limit μ of Δ; in particular, the conditions are trivially satisfied when G is abelian. Also, the embedding of a limit of a symmetric system on G is shown to hold under condition (1) as above.

1. Introduction

The central limit problem for group-valued independent random variables (X_j) concerns the limiting behaviour of the corresponding sequence of partial sums $S_n = \Sigma_{j=1}^n X_j$. In the classical case of real valued random variables the problem was solved by P. Levy, who found all possible limits of the sequence of normed sums $\{T_n = (1/b_n)S_n - a_n\}$ (where $b_n \in \mathbb{R}_+^*$, and $a_n \in \mathbb{R}$) under the additional hypothesis that X_j's are identically distributed. Subsequently, $S_n = \Sigma_{j=1}^{n_i} X_{ij}$ was considered, where $(X_{ij})_{i\in\mathbb{N},j=1}^{n_i}$ $(n_i \to \infty)$ is a triangular system of random variables which is infinitesimal in the sense that (X_{ij}) converges to zero as $i \to \infty$, uniformly in j.

By the classical Khintchine–Levy theory every limit of an infinitesimal triangular system on $\mathbb{R}$ is infinitely divisible and hence it is embeddable in a continuous

Received March 16, 1997 and in revised form January 8, 1998

one-parameter convolution semigroup. The same was considered on more general abelian groups, i.e. on divisible locally compact second countable groups by Parthasarathy et al. (cf. [PRV]) and the infinite divisibility of the limit was shown. Ruzsa eliminated the second countability condition and also generalised the above for all Banach spaces (cf. [R2]). A similar theorem was also proved by Gangolli for certain symmetric spaces (cf. [G]). A result of Carnal shows that infinite divisibility of limits holds for commutative infinitesimal triangular systems on compact groups (cf. [C]). An analogous result by Neuenschwander shows that the limit of such a system on a simply connected step-2 nilpotent group is embeddable in a continuous one-parameter semigroup if each X_{ij} is symmetric (cf. [N1]). The present author generalised the above to all connected nilpotent Lie (algebraic) groups provided each X_{ij} is symmetric or the measure corresponding to the limit has 'full' (algebraic) support (cf. [S3]). A recent result of Neuenschwander also shows the embeddability of the limit of such a system on any discrete subgroups of simply connected nilpotent groups (cf. [N2]). We refer the reader to [He2] for an exposition of the techniques and results on the problem for various classes of groups like abelian groups, compact groups and maximally aperiodic groups; see also [H] for the study on totally disconnected compact groups and some examples. We also refer the reader to [R1, R2, RS] for techniques based on the theory of Hungarian semigroups of probability measures on abelian groups and [S3] for generalisations to nonabelian groups, which will be used extensively here.

Here we study (in the measure-theoretic set-up) infinitesimal divisibility and embeddability of the limits of commutative infinitesimal triangular systems on more general locally compact groups G under certain conditions, such as when the system is symmetric or when G is totally disconnected.

Let S be a Hausdorff semigroup with identity e. A triangular system $\Delta = \{a_{ij} \in S \mid i \in \mathbb{N}, 1 \leq j \leq n_i, n_i \to \infty\}$ is said to be **commutative** if for all $i \in \mathbb{N}$, $a_{ij}a_{ik} = a_{ik}a_{ij}$ for all $1 \leq j, k \leq n_i$, **infinitesimal in** S if for every neighbourhood U of e in S, there exists an $i_0 \in \mathbb{N}$ such that $a_{ij} \in U$ for all $i > i_0$ and $1 \leq j \leq n_i$ (that is, a_{ij} converges to e as $i \to \infty$, uniformly in j) and **convergent** if the sequence of row-wise product $\{a_i = \Pi_{j=1}^{n_i} a_{ij}\}$ converges in S; if μ is its limit then we say that Δ **converges** to μ. For a subset A of S and $s \in S$ a decomposition as $s = s_1 \cdots s_n$, for some $n \in \mathbb{N}$, where $s_i \in A$ and $s_i s_j = s_j s_i$ for all i, j, is called an A**-decomposition of** s. An element $s \in S$ is said to be **infinitesimally divisible** if s has a U-decomposition for every neighbourhood U of e in S. For any $s \in S$, let T_s denote the set of all divisors (two sided factors) of s, i.e. $T_s = \{t \in S | \ tr = rt = s$ for some $r \in S\}$.

Let G be a locally compact group and let $M^1(G)$ be the topological semigroup of all Borel (regular) probability measures with weak topology and the convolution operation. For a $g \in G$, let δ_g denote the point mass at g and let $g\lambda = \delta_g\lambda$ (also, $\lambda g = \lambda\delta_g$) for any $\lambda \in M^1(G)$. A triangular system $\Delta = (\mu_{ij})_{i\in\mathbb{N},j=1}^{n_i}$ of probability measures on G is said to be infinitesimal if for any $\epsilon > 0$ and any neighbourhood U of the identity e in G, there exists $N \in \mathbb{N}$ such that for all $i \geq N$, $\mu_{ij}(U) > 1 - \epsilon$ for all j. (Note that this definition is the same as the infinitesimality in the semigroup $S = M^1(G)$ defined earlier; it is also equivalent to the infinitesimality of the corresponding system of random variables). A measure $\mu \in M^1(G)$ is said to be **infinitely divisible** (resp. **weakly infinitely divisible**) if for every $n \in \mathbb{N}$, there exists $\mu_n \in M^1(G)$ such that $\mu_n^n = \mu$ (resp. $\mu_n^n x_n = \mu$ for some $x_n \in G$); and it is said to be **embeddable** if there exists a continuous one-parameter semigroup $\{\mu_t\}_{t\geq 0}$ in $M^1(G)$ such that $\mu_1 = \mu$. Let $M(G)$ be the complex Banach algebra of all bounded complex measures on G with the usual norm (cf. [He2], pp. 31). For $\lambda \in M(G)$, the **adjoint of** λ, denoted by $\tilde{\lambda}$, is defined as $\tilde{\lambda}(B) = \overline{\lambda(B^{-1})}$, for all Borel sets B (recall that $B^{-1} = \{g^{-1} | \ g \in B\}$). We call a measure $\lambda \in M(G)$ **normal** (resp. **symmetric**) if $\lambda\tilde{\lambda} = \tilde{\lambda}\lambda$ (resp. $\lambda = \tilde{\lambda}$). We call a triangular system $\Delta = (\mu_{ij})_{i\in\mathbb{N},j=1}^{n_i}$ in $M^1(G)$ **normal** (resp. **symmetric**) if μ_{ij} is normal (resp. symmetric) for all i and j. Obviously, any symmetric measure (resp. triangular system) is normal.

Throughout the paper, G will be a locally compact (Hausdorff) topological group and Z will denote its center. Also G^0 will be the connected component of the identity in G. For a $\lambda \in M^1(G)$, let $G(\lambda)$ denote the smallest closed subgroup containing supp λ in G and let $N(\lambda)$, $Z(\lambda)$ denote respectively the normaliser and the centraliser of $G(\lambda)$ in G. For any $\lambda \in M^1(G)$, let $I_\lambda = \{x \in G | \ x\lambda x^{-1} = \lambda\}$ and $I(\lambda) = \{x \in G | \ x\lambda = \lambda x = \lambda\}$. Clearly, $I(\lambda)$ is compact and it is called the invariance group of λ. For a compact subgroup H in G, let ω_H be the Haar measure of H and let $M_H^1(G) = \omega_H M^1(G)\omega_H$. Then $M_H^1(G)$ is a closed subsemigroup of $M^1(G)$ with identity ω_H. For a $\lambda \in M^1(G)$ and any $x \in G$, a measure of the form λx (resp. $x\lambda$) is said to be a **right** (resp. **left**) **translate** of λ.

A group G is said to be **almost periodic** if all its finite dimensional irreducible unitary representations seperate points in G.

THEOREM 1.1: *Let G be a first countable locally compact group and let Δ be a commutative infinitesimal triangular system converging to μ. If either*

(a) *G is totally disconnected or*

(b) *if* Δ *is symmetric,*

then μ *is infintesimally divisible in* $M^1_H(G(\mu))$ *for some compact subgroup* $H \subset I(\mu)$. *Moreover, if* $G(\mu) \subset L$, *where* L *is a compact extension of a closed solvable normal subgroup or* L *is an almost periodic group, then* μ *is embeddable.*

A measure μ is said to be a **Poisson measure** if $\mu = \exp_H \nu$, where $\nu = \gamma(\lambda - \omega_H)$ for some $\gamma \in \mathbb{R}_+$ and $\lambda \in M^1_H(G)$, for a compact subgroup H of G (cf. [H], 3.2.1). By a result of Martin Löf any embeddable measure on a discrete group is a Poisson measure.

THEOREM 1.2: *Let* G *be any discrete group and let* Δ *and* μ *be as above. Suppose that one of the following holds:*

(1) $G(\mu)$ *is a finite extension of a solvable group;*

(2) $G(\mu)$ *is a linear group over a locally compact field of characteristic zero;*

(3) Δ *is normal.*

Then μ *is a Poisson measure.*

Theorems 1.1 and 1.2 generalise Theorem 1 of [N1], Theorem 1 of [N2], and also Theorem 1.2 of [S3] in the p-adic group case.

Remark: Theorem 1.1 also holds for Lie projective groups without the condition of first countability; this can be seen from the proof, using Proposition A.2 (see Appendix). If G is a totally disconnected Lie projective group, then it is a projective limit of discrete groups and hence, in the notation as above, when Δ is normal then μ is embeddable.

We also mention here the following corollaries of Theorem 4.2, which is a technical result proved in section 4.

COROLLARY 1.3: *Let* G *be a real almost algebraic group, i.e. a subgroup of finite index in an algebraic group. Let* Δ *and* μ *be as in Theorem 1.2. Suppose that* $Z(\mu)/Z$ *is compact, where* $Z(\mu)$ *is the centraliser of* $\operatorname{supp}\mu$. *Then* μ *is embeddable if* G *is connected nilpotent. If either* (1) G *is nilpotent or* (2) G *is a compact extension of a closed solvable normal subgroup and* Δ *is normal, then there exists an* $x \in G$ *such that* $x\mu$ *is embeddable.*

In the above corollary, if the closed subgroup generated by $\operatorname{supp}\mu$ is Zariski dense in G then $Z(\mu) = Z$. Corollary 1.3 is a generalisation of Theorem 1.2 in [S3]. Since G is almost algebraic, Z/Z^0 is finite and hence, from the hypothesis, $Z(\mu)/Z^0$ is compact. But one knows also that $I_\mu/Z(\mu)$ is compact (cf. [D], Corollary 2.5 or [DM], Theorem 3.2). This implies that I^0_μ/Z^0 is compact. The corollary follows from Theorem 4.2 and the remark following it, together with this observation.

The following corollary generalizes a theorem of Parthasarathy et al. (cf. [PRV] and also a theorem of Ruzsa (cf. [R2]).

COROLLARY 1.4: *Let G be a locally compact abelian group and let Δ and μ be as above. Then μ is infinitely divisible and there exists $x \in G^0$ such that $x\mu$ is embeddable; if, further, G^0 is arcwise connected, then μ itself is embeddable.*

The Corollary follows from Theorem 4.2 and Remark (1) following it, as any abelian group G is Lie projective and $G^0 = Z^0 = I_\mu^0$. The corollary can be generalised to first countable central groups up to a certain extent, as in this case one can get weak infinite divisibility and shift embeddability of μ. (A locally compact group G is called a **central group** if it is a compact extension of its center.)

The following result is about special triangular systems, i.e. the sequence of convolution powers. For previously known results see [N] or [S1, S2].

THEOREM 1.5: *Let G be a locally compact group and let $\{\nu_i\}$ be a sequence converging to δ_e. Suppose that $\{\nu_i^{k_i}\}$ converges to μ, for some unbounded sequence $\{k_i\}$. Assume also that $I_\mu^0/(I_\mu^0 \cap Z)$ is compact. Then μ is weakly infinitely divisible. If further I_μ^0/Z^0 is compact, then $x\mu$ is embeddable for some $x \in I_\mu^0$. If $I_\mu^0 = Z^0$, then μ is infinitely divisible and it is embeddable if Z^0 is arcwise connected.*

In the above Theorem, in particular if G is totally disconnected, then for any $\mu \in M^1(G)$, $I_\mu^0 = Z^0 = \{e\}$ and hence, for $\{\nu_i\}$ and μ as above, we get that μ is embeddable.

In the literature, the infinitesimality was considered in the neighbourhoods of identity perhaps because in the classical case of $\mathbb{R}^n$ there are no nontrivial compact subgroups. But in the general case of locally compact groups, we can consider infinitesimality in any neighborhoods of ω_K for a compact subgroup K. A triangular system Δ is said to be **ω_K-infinitesimal** if given a neighbourhood U of ω_K there exists $N \in \mathbb{N}$, such that for all $i \geq N$, $\mu_{ij} \in U$ for all j. Clearly, a measure μ which is embeddable in $\{\mu_t\}_{t\geq 0}$ is ω_K-infinitesimal where $\omega_K = \mu_0$. It turns out that all the results stated above hold if we replace infinitesimality by ω_K-infinitesimality. Moreover, we have the following:

THEOREM 1.6: *Let G be any group and let K be any compact open subgroup of G. Let Δ be any commutative ω_K-infinitesimal triangular system converging to μ. Suppose also that one of the following holds:* (1) *G is a closed subgroup of*

$GL(n, \mathbb{Q}_p)$, (2) Δ is symmetric or (3) G is totally disconnected and Δ is normal. Then μ is embeddable.

Perhaps a more natural generalisation of infinitesimality would be K-infinitesimality for a compact subgroup K of G defined as follows: a triangular system Δ on G is said to be K-**infinitesimal** if given $\epsilon > 0$, there exists a neighbourhood U of K and $N \in \mathbb{N}$, such that for all $i \geq N$, $\mu_{ij}(U) > 1 - \epsilon$ for all j. All the statements from Theorem 1.1 to Theorem 1.5 are valid if we assume that Δ is K-infinitesimal, instead of infinitesimal, with the additional condition that $K \subset I(\mu)$. In particular, we state the following, without a proof.

THEOREM 1.7: *Let G be a locally compact group and $\{\nu_i\}$ be a sequence such that $\{\nu_i^{k_i}\}$ converges to μ for some unbounded sequence $\{k_i\}$. If either (1) G is totally disconnected and $\nu_i \to \nu$ such that $G(\nu)$ is compact or (2) each ν_i is symmetric, then there exists $x \in G$ such that $x\mu$ is embeddable.*

It may be noted that the condition that $G(\nu)$ is compact is the same as saying that the triangular system of convolution powers $\{\mu_{ij} \mid \mu_{ij} = \nu_i, 1 \leq j \leq k_i\}$ is K-infinitesimal where $K = G(\nu)$.

In section 2, we construct semigroups S in which the limit μ of a given triangular system Δ is infinitesimally divisible. In section 3, we construct partial homomorphisms on the factor set of μ under certain conditions which would enable us to prove the embedding of μ or its shift in section 4. We also prove a more general result in section 4 (see Theorem 4.2).

ACKNOWLEDGEMENT: It is a pleasure to thank the Hebrew University of Jerusalem, in particular, Prof. Shahar Mozes, for hospitality while this work was done. I would like to thank Prof. S. G. Dani for discussions and help in proving Proposition A.1. I am also grateful to him for his comments on a preliminary version of the manucript. Thanks are also due to the referee whose comments lead to improvement of the manuscript.

2. Infinitesimal divisibility

Let G be any locally compact group and let Δ be a commutative infinitesimal convergent triangular system with limit μ. If G is totally disconnected or if Δ is symmetric, we first construct an abelian semigroup S of $M^1(G)$ such that the limit μ belongs to S, μ is infinitesimally divisible in S and T_μ in S is compact. We also construct such a semigroup under more general condition (see Proposition 2.9). For this, we use a method as in [S3] combined with some new results based on infinitesimality and concentration functions (see Lemma 2.1 and Theorem 2.4).

LEMMA 2.1: *Let G be any locally compact group and $\mu \in M^1(G)$. Let F be any compact subset of G and let $A = \{\lambda \in M^1(G) | \lambda(F) \geq \delta\}$ for some fixed $\delta > 0$. Let $\{\lambda_\alpha\}$, $\{\nu_\alpha\}$, $\{\mu_\alpha\}$ be nets in $M^1(G)$ such that $\{\lambda_\alpha\} \subset A$ and $\lambda_\alpha \nu_\alpha = \mu_\alpha \to \mu$. Then $\{\lambda_\alpha\}$ is relatively compact. In particular, $T_\mu \cap A$ is compact. Also, $T_\mu \cap \overline{U}$ is compact for small neighbourhoods U of δ_e in $M^1(G)$.*

Proof: There exists a net $\{x_\alpha\}$ in G such that $\{\lambda_\alpha x_\alpha\}$ is tight (cf. [He2], Theorem 1.2.21). That is, given an $\epsilon > 0$, there exists a compact set K such that $\lambda_\alpha x_\alpha(G \setminus K) \leq \epsilon$ for all α. Let $\epsilon < \delta$. Then $\lambda_\alpha(K x_\alpha^{-1} \cap F) \neq \emptyset$ and hence $x_\alpha \in F^{-1}K$ for each α. Thus $\{\lambda_\alpha\}$ is tight and therefore it is relatively compact (cf. [St]). In particular, $T_\mu \cap A$ is compact, as both T_μ and A are closed. The last assertion easily follows now as, if V is a compact neighbourhood of the identity e in G, then for a sufficiently small neighbourhood U of δ_e, $\lambda(V) \geq 1/2$, for all $\lambda \in U$. ∎

For a measure $\mu \in M^1(G)$ and $n \geq 1$ let $c_n(K) = \sup_{x \in G} \mu^n(Kx)$, for any compact subset K of G; c_n are called the **concentration functions of μ.**

We now note a Lemma which is a consequence of a result on concentration functions in [DS].

LEMMA 2.2: *Let G be a Lie group with finitely many connected components. Let R be the radical of G and suppose that the center of G^0/R is finite. Let $\lambda \in M^1(G)$ be such that the concentration functions of λ and $\tilde{\lambda}$ fail to converge to zero. Then there exist a continuous one-parameter subgroup ϕ and a compact subgroup L such that $G(\lambda) \subset \phi \times L$ and $\operatorname{supp} \lambda \subset xL$ for some $x = \phi(1)$.*

Proof: Since the concentration functions of λ do not converge to zero, under the condition on G as in the hypothesis as above, Theorem 3 of [DS] yields the following: there exist a closed subgroup $C \subset G$ and closed normal subgroups H, N of C such that $G(\lambda) \subset C$, N is simply connected and nilpotent, H/N is compact, $C/N = \phi' \times H/N$ for some (possibly trivial) one-parameter subgroup ϕ' and for every $x \in \operatorname{supp} \lambda$, $\operatorname{supp} \lambda \subset xH = Hx$ and the conjugation action of x on N is a contraction (cf. Theorem 3, [DS]). Since the concentration functions of $\tilde{\lambda}$ do not converge to zero and C/R' is compact, where R' is the radical of C, there exist a closed subgroup $\tilde{C} \subset C$ such that $G(\tilde{\lambda}) \subset \tilde{C}$ and closed normal subgroups $\tilde{H}$ and $\tilde{N}$ such that $\tilde{N}$ is simply connected and nilpotent, $\tilde{H}/\tilde{N}$ is compact, $\tilde{C}/\tilde{N} = \phi'' \times \tilde{H}/\tilde{N}$ for some (possibly trivial) one-parameter subgroup ϕ'' and for any $y \in \operatorname{supp} \tilde{\lambda}$, $\operatorname{supp} \tilde{\lambda} \subset y\tilde{H} = \tilde{H}y$ and the conjugation action of

y contracts $\tilde{N}$. Since $C/N = \phi' \times H/N$, and x^{-1} contracts $\tilde{N}$, it easily follows that $\tilde{N} \subset N$. But $N \cap \tilde{N}$ is trivial as both x and x^{-1} contract $N \cap \tilde{N}$ and hence $\tilde{N}$ is trivial. This implies that $G(\lambda) = G(\tilde{\lambda}) \subset \phi \times L$, where $L = \tilde{H}$ is compact, $\phi_t = \phi''_{-t}$ and $\operatorname{supp} \lambda \subset xL = Lx$, where x can be chosen such that $x = \phi(1)$. ∎

We recall that a locally compact group G is said to be **almost connected** if G/G^0 is compact. Given a locally compact group G, G/G^0 is totally disconnected and hence zero-dimensional and so G admits open almost connected subgroups.

PROPOSITION 2.3: *Let G be an almost connected group. Suppose that $\lambda \in M^1(G)$ is such that the concentration functions of both λ and $\tilde{\lambda}$ fail to converge to zero. Let $x \in \operatorname{supp} \lambda$. Then $\{\lambda^n x^{-n}\}$ and $\{x^{-n}\lambda^n\}$ are tight and all their limit points are respectively right and left translates of ω_H for some fixed compact subgroup H. Also, $\operatorname{supp} \lambda \subset Hx = xH$.*

Proof: Since G is almost connected, G is Lie projective. There exist compact normal subgroups K_α such that $\cap_\alpha K_\alpha = \emptyset$ and G is a projective limit of Lie groups $G_\alpha = G/K_\alpha$, with finitely many connected components. Now the assertion holds on G if and only if the images of $\{\lambda^n x^{-n}\}$ and $\{x^{-n}\lambda^n\}$ on each G_α are tight and all its limit points are respectively right and left translates of an idempotent ω_{H_α}, for some H_α. Therefore without loss of generality, we may assume that G is a Lie group with finitely many connected components. Then by Proposition A.1 (see Appendix), we may further assume that G satisfies the condition in the hypothesis of Lemma 2.2. Then by the Lemma, there exists a (possibly trivial) continuous one-parameter subgroup ϕ and a compact subgroup L such that $G(\lambda) \subset \phi \times L$ and $\operatorname{supp} \lambda \subset yL$, where $y = \phi(1)$. Therefore, $\{\lambda^n x^{-n}\}$ and $\{x^{-n}\lambda^n\}$ are contained in $M^1(L)$ and hence they are tight. Let $\nu = y^{-1}\lambda = \lambda y^{-1}$. Then $\operatorname{supp} \nu \subset L$ and $\overline{\{\nu^n\}}$ is a compact semigroup. There exists a unique idempotent ω_H in this semigroup and all the limit points of $\{\nu^n\}$ are of the form $a\omega_H = \omega_H a$ (for some $a \in G$), i.e. two sided translates of ω_H. Since $x \in \operatorname{supp} \lambda$, $x = yl = ly$ for some $l \in L$. Then for each n, $\lambda^n x^{-n} = \lambda^n y^{-n} l^{-n} = \nu^n l^{-n}$, and similarly, $x^{-n}\lambda^n = l^{-n}\nu^n$. Therefore, the limit points of $\{\lambda^n x^{-n}\}$ (resp. $\{x^{-n}\lambda^n\}$) are right (resp. left) translates of ω_H. Clearly, $\operatorname{supp} \lambda \subset xH = Hx$. This completes the proof. ∎

The following Theorem plays an important role in our construction.

THEOREM 2.4: *Let G be any locally compact group and let $\mu \in M^1(G)$. Let V be a neighbourhood of the identity in G contained in an almost connected* *subgroup of G. Let $\lambda \in M^1(G)$ be such that $\lambda(V) > 0$. Suppose that $\lambda^n \in T_\mu$ for all n. Then $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$, where $x \in V \cap I_\mu$.*

Proof: Since $\lambda^n \in T_\mu$ for all n, there exist sequences $\{x_n\}$ and $\{y_n\}$ in G, such that $\{\lambda^n x_n\}$ and $\{y_n\lambda^n\}$ are tight (cf. [He], Theorem 1.2.21). That is, given $\epsilon > 0$, there exists a compact set K such that $\lambda^n x_n(K) > 1-\epsilon$ and $y_n\lambda^n(K) > 1-\epsilon$, for all n. Therefore the concentration functions of both λ and $\tilde{\lambda}$ fail to converge to zero. Hence by Theorem 2.18 of Jaworski et al. [JRW], either $G(\lambda)$ is compact or $G(\lambda)/N_\lambda$ is infinite cyclic, where N_λ is the smallest closed normal subgroup of $G(\lambda)$ such that $\operatorname{supp}\lambda \subset xN_\lambda$, for some $x \in G(\lambda)$.

Since $\lambda(V) > 0$, x as above can be chosen to be contained in V. If G is totally disconnected then x generates a compact subgroup, say G_x. But then $G(\lambda) \subset G_x N_\lambda$, which implies that $G(\lambda)/N_\lambda$ is compact. Hence from the reduction as above $G(\lambda)$ is compact.

Since G/G^0 is totally disconnected and the concentration functions of the image $\overline{\lambda}$ of λ on G/G^0 do not converge to zero, from the above arguments, it follows that $G(\overline{\lambda})$ is compact. This implies that $\overline{G(\lambda)G^0}/G^0$ is compact.

Let $M = \overline{G(\lambda)G^0}$. Then M is an almost connected group. Let $x \in \operatorname{supp}\lambda$. Then by Proposition 2.3, $\{\lambda^n x^{-n}\}$ (resp. $\{x^{-n}\lambda^n\}$) is tight and if ν (resp. ν') is any limit point of it then $\nu = \omega_H y$ for some $y \in \operatorname{supp}\nu$ (resp. $\nu' = y'\omega_H$ for some $y' \in \operatorname{supp}\nu'$) for some compact subgroup H such that $\operatorname{supp}\lambda \subset xH = Hx$. Therefore ω_H is a limit point of $\{\lambda^n x^{-n} y^{-1}\}$ (resp. $\{y'^{-1}x^{-n}\lambda^n\}$). Since $\lambda^n \in T_\mu$ for all n, it follows that $\omega_H \in T_\mu$ and hence $H \subset I(\mu)$. Since $\operatorname{supp}\lambda \subset xH = Hx$, $\lambda\mu = x\mu = \mu\lambda = \mu x$ and hence $x\mu x^{-1} = \mu$. This implies that $xI(\mu)x^{-1} = I(\mu)$ and $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$. This completes the proof. ∎

Let $J_\mu = \{\lambda \in M^1(G) \mid \lambda\mu = \mu\lambda = \mu\}$. Clearly, J_μ is a compact semigroup and for any $\lambda \in M^1(G)$, $\lambda \in J_\mu$ if and only if $\operatorname{supp}\lambda \subset I(\mu)$.

The proof of Lemmas 2.5 and 2.6 are similar to that of Lemma 3.1 in [S3].

LEMMA 2.5: *Let G and μ be as above. Let U and W be neighbourhoods of J_μ in $M^1(G)$, with $W \subset U$. Let V be a relatively compact neighbourhood of e in G such that $VI(\mu)G^0$ is contained in an almost connected subgroup of G. Suppose that there exists a $\delta > 0$ such that $\lambda(VI(\mu)) > \delta$ for all $\lambda \in U$. Then there exists an n such that for $m \geq n$, μ cannot be expressed as $\mu = \lambda_1 \cdots \lambda_m$ with $\lambda_1, \ldots, \lambda_n \in T_\mu \cap U \setminus \overline{V}W$ and λ_i's commuting with each other.*

Proof: Here, $\overline{U} \subset \{\lambda \mid \lambda(\overline{V}I(\mu)) \geq \delta > 0\}$ and since $\overline{V}I(\mu)$ is compact, by Lemma 2.1, $T_\mu \cap \overline{U}$ and hence $T_\mu \cap \overline{U} \setminus \overline{V}W$ is compact. Let $\lambda \in \overline{U} \setminus \overline{V}W$. Suppose, if possible, $\lambda^n \in T_\mu$ for all n. Then by Theorem 2.4, $\operatorname{supp}\lambda \subset xI(\mu) =$

$I(\mu)x$. Since $\lambda(\overline{V}I(\mu)) \geq \delta > 0$, this implies that $x \in \overline{V}I(\mu)$, and hence $\lambda \in \overline{V}J_\mu$, which is a contradiction since $\lambda \notin \overline{V}W$. Thus for any $\lambda \in T_\mu \cap \overline{U} \setminus \overline{V}W$, there exists $n(\lambda)$ such that $\lambda^{n(\lambda)} \notin T_\mu$. Let V' be a neighbourhood of $\lambda^{n(\lambda)}$ such that $V' \cap T_\mu = \emptyset$ and let V_λ be a neighbourhood of λ such that $V_\lambda^{n(\lambda)} \subset V'$. Now since $T_\mu \cap \overline{U} \setminus \overline{V}W$ is compact, there exist $\lambda_1, \ldots, \lambda_l \in T_\mu \cap \overline{U} \setminus \overline{V}W$ such that $T_\mu \cap \overline{U} \setminus \overline{V}W \subset \bigcup_{i=1}^l V_{\lambda_i}$. Choose $n = \sum_{i=1}^l n(\lambda_i)$. Now, the assertion clearly holds for this n. ∎

LEMMA 2.6: *Let G be a totally disconnected locally compact group and let Δ be a commutative infinitesimal triangular system converging to μ. Then given a neighbourhood U of J_μ in $M^1(G)$, μ has a U-decomposition.*

Proof: Without loss of generality we may assume that

$$U = \{\lambda \in M^1(G) | \lambda(VI(\mu)) > \delta\}$$

for some relatively compact neighbourhood V of e in G such that the condition in the above Lemma holds, and for some fixed $\delta > 0$. Let W and W' be neighbourhoods of J_μ and δ_e respectively such that $W' \subset W$ and $\overline{WW'} \subset U$. We apply the above Lemma to U, V and W, and let n be as in the conclusion of the Lemma. Let $\Delta = (\mu_{ij})_{i \in \mathbb{N}, j=1}^{n_i}$. Since Δ is infinitesimal, there exists i_0 such that $i \geq i_0$, $\mu_{ij} \in W'$ for all j.

We define sequences $\{x_{ik}\}_{k=1}^n$ and $\{y_i\}$ as follows: let $i > i_0$ be given and let $\{m_0, \ldots, m_n\}$ be defined inductively as follows: set $m_0 = 0$ and, after m_k is defined for a $k < n$, let m_{k+1} be the smallest m such that $\prod_{j=m_k+1}^m \mu_{ij} \notin \overline{V}W$ if $m_k < n_i$ and such an m exists, and $m_{k+1} = n_i$ if either of the conditions fails. For $1 \leq k \leq n$, let $l_k = m_{k-1}$ and $x_{ik} = \prod_{j=l_k+1}^{m_k} \mu_{ij}$, if $m_{k-1} < n_i$, and $x_{ik} = \delta_e$ otherwise. Let $y_i = \Pi_{j=m_n+1}^{n_i} \mu_{ij}$ if $m_n < n_i$, otherwise $y_i = \delta_e$. Clearly, all $x_{ik} \in \overline{V}WW' \subset \overline{V}U$ and either $y_i = \delta_e$ or all x_{ik} are outside $\overline{V}W$. Since all $x_{ik} \in T_{\mu_i}$ for each i, by Lemma 2.1, $\{x_{ik}\}$ is relatively compact for each k, and so is $\{y_i\}$. Therefore, passing to a subsequence and altering the notation suitably we may assume that $\{x_{ik}\}$ and $\{y_i\}$ converge to (say) x_k and y, in T_μ, respectively. Then all x_k's and y commute with each other and $x_1 \cdots x_n y = \mu$. Also, since $x_{ik} \in \overline{V}WW'$, for all i, $x_k \in \overline{VWW'} \subset \overline{V}U$ (as $\overline{V}$ is compact and $\overline{WW'} \subset U$). Here, since G is totally disconnected one can choose V to be an open compact group and then $\overline{V} = V$ and $VU = U$ and hence $x_k \in U$. If $y = \delta_e$, then the above is a U-decomposition of μ. If $y \neq \delta_e$, then the set $\{i\colon y_i = \delta_e\}$ cannot be cofinal. Hence $x_k = \lim x_{ik} \notin \overline{V}W$ for all $1 \leq k \leq n$, by Lemma 2.5, which is a contradiction to the choice of n. This proves the Lemma. ∎

LEMMA 2.7: *Let G be a locally compact group and let Δ and μ be as above. If Δ is symmetric then, for any neighbourhood U of J_μ in $M^1(G)$, μ has a U-decomposition.*

STEP 1: Let $N(I(\mu))$ be the normaliser of $I(\mu)$ in G. Since $I(\mu)$ is a compact normal subgroup in $N(I(\mu))$, there exists an open subgroup N of $N(I(\mu))$ containing $I(\mu)$ such that N is Lie projective. Let K_α be the compact normal subgroups of N such that N is the projective limit of the Lie groups $N_\alpha = N/K_\alpha$. Consider the group $N' = N/I(\mu)$. Then N' is the projective limit of the Lie groups $N'_\alpha = N/(I(\mu)K_\alpha)$. Let α be fixed. There exists a neighbourhood V'_α of the identity $\overline{e}$ in N'_α such that if $x \in V'_\alpha \setminus \{\overline{e}\}$, $x^2 \neq \overline{e}$. Let V_α be the inverse image of V'_α in N. Then V_α is open in N and hence in $N(I(\mu))$ and if $x \in V_\alpha$, such that $x^2 \in I(\mu)K_\alpha$, then $x \in I(\mu)K_\alpha$. Let V be a relatively compact neighbourhood of identity in G such that $V = K_\alpha V$ and $\overline{V}I(\mu) \cap N(I(\mu)) \subset V_\alpha$. Without loss of generality we may assume that $U = \{\lambda \mid \lambda(VI(\mu)) > \delta\}$ for some fixed $\delta > 0$. Then $K_\alpha U = U$.

STEP 2: Let W and W' be neighbourhoods of J_μ and δ_e respectively such that $W' \subset W$, and that $\overline{WW'} \subset U$. Let $\lambda \in T_\mu \cap \overline{U} \setminus K_\alpha W$ be a symmetric measure, and suppose if possible that $\lambda^n \in T_\mu$ for all n. Then by Theorem 2.4, $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$ and $x^2 \in I(\mu)$ for any $x \in \operatorname{supp}\lambda$. Since $\lambda(VI(\mu)) > \delta > 0$, this implies that $x \in VI(\mu)$, and by the choice of V in step 1, $x \in K_\alpha I(\mu)$. Therefore $\operatorname{supp}\lambda \subset yI(\mu)$, for some $y \in K_\alpha$, and hence $\lambda \in K_\alpha J_\mu$. This is a contradiction as $\lambda \notin K_\alpha W$. Now since $T_\mu \cap \overline{U}$ is compact, as in the proof of Lemma 2.5, one can choose n such that for $m \geq n$, $\mu \neq \lambda_1 \cdots \lambda_m$, for any mutually commuting symmetric measures $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1, \ldots, \lambda_n \in T_\mu \cap \overline{U} \setminus K_\alpha W$. Now using this and the fact that $K_\alpha \overline{WW'} \subset K_\alpha U = U$, one can easily prove, along the lines of the proof of the previous Lemma, that μ has a U-decomposition. ∎

Let S be a semigroup with identity e. Elements $s, t \in S$, are said to be **associates** if $s = s't = ts'$ and $t = t's = st'$ for some $s', t' \in S$. A subset A of S is said to be **associatefree** if $s, t \in A$ are associates, then $s = t$. An element s is said to be **bald** if for any idempotent $h \in S$ if $hs = sh = s$, then $h = e$.

PROPOSITION 2.8: *Let G be a locally compact first countable group and let Δ and μ be as above. Then if G is totally disconnected or if Δ is symmetric, then there exists a closed commutative subsemigroup S of $M^1_H(G(\mu))$ with identity ω_H such that $\mu \in S$, μ is bald and infinitesimally divisible in S, and T_μ (in S) is compact and associatefree.*

Proof: The first part of the proof is similar to Proposition 3.2 in [S3]. We give the whole proof here for the sake of completeness.

Given a neighbourhood U of J_μ, by Lemmas 2.6 and 2.7, μ has a U-decomposition. Since G is first countable, one can find a neighbourhood basis $\{U_k|\ k \in \mathbb{N}\}$ of J_μ such that $U_{k+1} \subset U_k$.

Now for $k \in \mathbb{N}$, let $\mu = s_1 \cdots s_{m(k)}$ be a U_k-decomposition of μ obtained as in Lemma 2.6; here $s_l = \lim_{i\to\infty} \Pi_{j\in A(i,l)} \mu_{k(i)j}$, where $\{k(i)\} \subset \mathbb{N}$ is a fixed sequence for all l and $A(i,l) \subset \{1, \ldots, n_i\}$ depends on i and l, and for fixed i, $A(i,l)$ are disjoint. Now using the subsystem $(\mu_{k(i)j})_{i\in\mathbb{N}, j\in A(i,l)}$, one can get a U_{k+1}-decomposition of $s_l = s_{l1} \cdots s_{lm(l)}$, such that $s_{ln}s_{pq} = s_{pq}s_{ln}$ for all l, n, p, q. Here, for all l and n, $s_{ln} = \lim_{i\to\infty} \Pi_{j\in A(i,l,n)} \mu_{(k+1)(i)j}$, where $\{(k+1)(i)\}$ is a fixed subsequence of $\{k(i)\}$ and $A(i,l,n) \subset A(i,l)$ depends on i, l, n. We continue this process.

For $k \in \mathbb{N}$, let M_k be the semigroup generated by $\{s_{k1}, \ldots, s_{km(k)}\}$ in $M^1(G)$, where $\mu = s_{k1} \cdots s_{km(k)}$ is the U_k-decomposition of μ obtained in the above manner. Then for all k, $M_k \subset M_{k+1}$, M_k is abelian and $\mu \in M_k$. Let $S_1 = \overline{\bigcup_k M_k}$. Then S_1 is an abelian semigroup containing μ. Also, each α in S_1 is a limit of U_k-decomposition in S_1 for small neighbourhoods U_k of J_μ. Let $\alpha \in S_1$ such that $I(\alpha) \subset I(\mu)$. Then $J_\alpha \subset J_\mu$. Arguing as in Lemmas 2.5–2.7 and using that $T_\alpha \cap \overline{U}$ is compact for a small neighbourhood U of J_μ, and that J_μ is a compact semigroup, one can show that α has a U-decomposition in S_1 for every neighbourhood U of J_μ.

Now let $J = J_\mu \cap S_1$. Then J is a nonempty compact abelian semigroup. Since $I(\mu)$ is compact, a simple calculation shows that, given any neighbourhood U_J of J in S_1, there exists a neighbourhood U of J_μ such that $U \cap S_1 \subset U_J$. Therefore μ and each $\alpha \in S_1$, such that $I(\alpha) \subset I(\mu)$, have a U_J-decomposition for every neighbourhood U_J of J in S_1.

Since J is a compact abelian semigroup, there exists a maximal idempotent $l = \omega_L$ in J, where L is some compact subgroup of $I(\mu)$. Then $J' = Jl$ is a compact abelian group and $S_2 = S_1 l$ is an abelian semigroup with identity l. Let $H = \{x \in G|$ such that $xl \in J'\}$. Then H is a compact subgroup contained in $I(\mu)$; let $h = \omega_H$. Clearly, $h\mu = \mu$, $J'h = Jh = \{h\}$. Now for any $x \in H$, and $\lambda \in S_2$, $x\lambda = xl\lambda = \lambda(xl) = \lambda lx = \lambda x$. This implies that $h\lambda = \lambda h$ for all $\lambda \in S_2$. Let $S = S_2 h = S_1 h$. Then S is an abelian semigroup, with identity h and $\mu \in S$. Let U be any neighbourhood of h. Since J is compact, there exists a neighbourhood W of J such that $Wh \subset U$. Now if $\mu = \lambda_1 \cdots \lambda_n$ is a W-decomposition in S_1, then each $h\lambda_i = \lambda_i h \in Wh \cap S_1 h \subset U \cap S$ for all i and

$h\lambda_1 \cdots h\lambda_n = h^n\lambda_1 \cdots \lambda_n = h\mu = \mu$. Thus μ has a U-decomposition in S for every neighbourhood U of h. Let $\alpha \in T_\mu$ in S; then $\alpha = \alpha' h$ for some $\alpha' \in S_1$ and $\alpha' hb = bh\alpha' = \mu$ for some b and hence $I(\alpha') \subset I(\mu)$. Now from the above, $\alpha' = \alpha'_1 \cdots \alpha'_n$ is a W-decomposition of α' in S_1 for a neighbourhood W of J. Then $\alpha = \alpha' h = \alpha'_1 h \cdots \alpha'_n h$ is a U-decomposition of α in S, where $Wh \subset U$. In particular, any $\alpha \in T_\mu$ is infinitesimally divisible in S.

Let U be a neighbourhood of h, such that $U = \{\nu \in M^1(G) |\, \nu(VH) > \delta > 0\}$ for some relatively compact neighbourhood V of e in G and some $\delta > 0$, such that if G is totally disconnected then V is any open compact group normalised by $I(\mu)$ and if Δ is symmetric then V is chosen as in the proof of Lemma 2.7.

Let $\lambda, \nu \in T_\mu$ (in S) be associates. Then there exists λ', $\nu' \in S$ such that $\lambda = \lambda'\nu$ and $\nu = \nu'\lambda$. Therefore $\operatorname{supp} \lambda'\nu' \subset I(\lambda) \subset I(\mu)$. Then $\operatorname{supp} \lambda' \subset xI(\mu) = I(\mu)x$ for some $x \in \operatorname{supp} \lambda'$. Let $\lambda' = u_1 \cdots u_n$ be a U-decomposition of λ'. Then $\operatorname{supp} u_i \subset x_i I(\mu) = I(\mu)x_i$, $x_i \in V$. Hence if G is totally disconnected, V is an open compact subgroup normalised by $I(\mu)$ and hence $\operatorname{supp} \lambda' \subset VI(\mu)$. Since this holds for all such V which form a neighbourhood basis of identity in G, $\operatorname{supp} \lambda' \subset I(\mu)$. If Δ is symmetric, then each element of S is symmetric and hence $x_i^2 \in I(\mu)$ for all i. From the choice of V, we get that $x_i \in I(\mu)K_\alpha = K_\alpha I(\mu)$ for all i and hence $\operatorname{supp} \lambda' \subset I(\mu)K_\alpha$, where $K_\alpha \subset V$ is a compact subgroup as in Lemma 2.7. Since V forms a neighbourhood basis of identity in G, $\operatorname{supp} \lambda' \subset I(\mu)$.

Thus in both the cases, $\operatorname{supp} \lambda' \subset I(\mu)$. But since $\lambda' = \alpha' h$ for some $\alpha' \in S_1$, $\operatorname{supp} \alpha' \subset I(\mu)$ and hence $\alpha' \in J_\mu \cap S_1 = J$. This implies that $\lambda' \in Jh = \{h\}$ and hence $\lambda = \nu$. That is, T_μ in S is associatefree.

Now let $a \in S$ be an idempotent such that $a\mu = \mu a = \mu$. Then $\operatorname{supp} a \subset I(\mu)$. Then arguing as above, we get that $a = ah = h$. This proves that μ is bald in S.

We now show that T_μ in S is compact. Let U and V be as above. Let W be a neighbourhood of h such that $\overline{WW} \subset U$. Now if $\lambda \in T_\mu \cap U \setminus W$ and $\lambda^n \in T_\mu$ for all n, then $\operatorname{supp} \lambda \subset xI(\mu) = I(\mu)x$, for some x (cf. Theorem 2.4). Since λ is infinitesimally divisible, arguing as above one can get that in both the cases, $\operatorname{supp} \lambda \subset I(\mu)$ and hence $\lambda = h$, a contradiction as $\lambda \notin W$. Now if $\nu \in T_\mu$, $T_\nu \subset T_\mu$, and hence by the above argument, as in Lemma 2.5, there exists an n such that for $m \geq n$, $\nu \neq \lambda_1 \ldots \lambda_m$ in S, if λ_i's commute and $\lambda_1, \ldots, \lambda_n \in T_\mu \cap U \setminus W$.

Let $\{\lambda_k\} \subset T_\mu$ (in S). Since for each k, $T_{\lambda_k} \subset T_\mu$ and λ_k is infinitesimally divisible, using the above and arguing as in Lemma 2.6 (with U and W as above), there exists an n such that $\lambda_k = \lambda_{k1} \cdots \lambda_{kn}$ is a U-decomposition for each k. This implies that $\lambda_k \in U^n$, i.e. $\lambda_k(VH)^n > \delta^n$ for each k. Since H is compact and V is relatively compact, this together with Lemma 2.1 implies that $\{\lambda_k\}$ is relatively compact. This implies that T_μ in S is compact.

It remains to show that $S \subset M^1_H(G(\mu))$. Since S is generated by T_μ, it is enough to show that the elements of T_μ are supported on $G(\mu)$. Clearly, $H \subset G(\mu)$. Let $\alpha \in T_\mu$. Then $\operatorname{supp}\alpha \subset xG(\mu) = G(\mu)x$ for any $x \in \operatorname{supp}\alpha$. Since α is infinitesimally divisible in S, $\overline{x} = xG(\mu)$ is infinitesimally divisible in $N(\mu)/G(\mu)$. That is, $\overline{x} = \overline{x}_1 \cdots \overline{x}_n$ for any neighbourhood V of identity $\overline{e}$ in $N(\mu)/G(\mu)$, where $N(\mu)$ is the normaliser of $G(\mu)$. Now if G is totally disconnected then open compact subgroups V form a neighbourhood basis of $\overline{e}$ and hence $\overline{x} \in V$ for all such V, therefore $\overline{x} = \overline{e}$, i.e. $\operatorname{supp}\alpha \subset G(\mu)$. Now if Δ is symmetric then the elements of S are symmetric and hence $\overline{x}_i^2 = \overline{e}$ for all i. Let L be an open Lie projective subgroup of $N(\mu)/G(\mu)$; let K_γ be the compact normal subgroups of L such that L is a projective limit of L/K_γ. Then as in the proof of Lemma 2.7, one can choose small neighbourhoods V_γ such that if $\overline{x}_i \in V_\gamma$ such that $\overline{x}_i^2 = \overline{e}$, then $\overline{x}_i \in K_\gamma$ for each γ. Therefore $\overline{x} \in K_\gamma$ for all γ and hence $\overline{x} = \overline{e}$, i.e. $\operatorname{supp}\alpha \subset G(\mu)$. In fact, the above gives that for any $\alpha \in T_\mu$, elements of T_α are supported on $G(\alpha)$. This completes the proof. ∎

For a semigroup S with identity e, an element $s \in S$ is said to be **weakly infinitesimally divisible** if given any neighbourhood U of e in S there exist $s_1, \ldots, s_n \in U$ and an invertible element $u \in S$ such that $s = us_1 \cdots s_n$, s_i's commute with each other and also with u. The following proposition shows the existence of an abelian semigroup containing the limit μ of a given triangular system under a more general set up such that μ is weakly infinitesimally divisible.

PROPOSITION 2.9: *Let G be a locally compact first countable group and let Δ be a commutative infinitesimal triangular system converging to μ. Suppose that $I^0_\mu/(I^0_\mu \cap Z)$ is compact. Then there exist a Hausdorff abelian semigroup S in $M^1(G)$ with identity ω_H and an equivalence relation $\approx$ on S such that $\mu \in S$, and if $\pi\colon S \to S' = S/\approx$ is the natural projection then $\pi(\mu)$ is bald and infinitesimally divisible in S' and $T_{\pi(\mu)}$ is compact. Moreover, for any $\alpha \in T_\mu$ and any neighbourhood U of ω_H in S, there exist $\lambda_1, \ldots, \lambda_n \in U$ and an invertible element $u \in S$ such that $\alpha = \lambda_1 \cdots \lambda_n u$ where $u = \delta_x * \omega_H$ for some $x \in I^0_\mu$.*

Proof: Let H be an open subgroup of G such that H/G^0 is compact. Then H is Lie projective. Let $\{K_n\}$ be a decreasing sequence of compact normal subgroups such that each H/K_n is a Lie group and $\bigcap_n K_n = \{e\}$, the trivial subgroup. Let $H' = H \cap I_\mu$. Then H' is an open subgroup in I_μ and it is a projective limit of $H'_n = H'/K'_n$ where $K'_n = I_\mu \cap K_n$. Then K_nG^0 is open in G and $K'_nI^0_\mu$ is open in I_μ. Let V be an open relatively compact neighbourhood of e such that $K_mV = V$, $V \subset K_mG^0$ and $V \cap I_\mu \subset K'_m I^0_\mu$ for all m.

We fix a δ such that $0 < \delta < 1$, and let $U = \{\lambda \in M^1(G)| \ \lambda(VI(\mu)) > \delta\}$ be a neighbourhood of J_μ. By Lemma 2.1, $T_\mu \cap \overline{U}$ is compact. Let $\lambda \in U$ be such that $\lambda^n \in T_\mu$ for all n. Then by Theorem 2.4, $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$ for some $x \in V \cap I_\mu \subset K'_m I^0_\mu$. Then $G(\lambda) \subset K'_m I^0_\mu I(\mu)$. Since $I(\mu)$ is compact, without loss of generality we may assume that $I(\mu) \subset H$.

Now we can define an equivalence relation '$\sim$' as follows: for $\nu, \nu' \in M^1(G)$, $\nu \sim \nu'$ if $\nu = z\nu'$ for some $z \in I^0_\mu \cap Z$. Then $M'(G) = M^1(G)/\sim$ is a semigroup and the corresponding map $\pi_1 \colon M^1(G) \to M'(G)$ is a continuous open homomorphism. Since $I^0_\mu/(I^0_\mu \cap Z)$ is compact, $\pi_1(\lambda)$ generates a compact semigroup, in fact since $\pi_1(\lambda) = \pi_1(\delta_x)\pi_1(\nu) = \pi_1(\nu)\pi_1(\delta_x)$ where $\operatorname{supp}\nu \subset I(\mu)$, $x \in K'_m I^0_\mu I(\mu)$, we have that $\pi_1(\delta_x)$ generates a compact group. Also $T_{\pi_1(\mu)} \cap \pi_1(\overline{U})$ is compact. Let $K' = \pi_1(I^0_\mu)$, which is a compact group. Let W and W' be respectively neighbourhoods of J_μ and δ_e such that $W' \subset W$, $K'_m W = W$ and $\overline{WW'} \subset U$. Let λ be such that $\pi_1(\lambda) \in K'\pi_1(U) \setminus K'\pi_1(W)$. Then there exists an n such that $\pi_1(\lambda^n) \notin T_{\pi_1(\mu)}$ as, otherwise for every n, $\pi_1(\lambda)^n \in T_{\pi_1(\mu)}$ then $\lambda^n \in T_\mu$, and since $\pi_1(\lambda) \in K'\pi_1(U)$, this implies that $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$ for some $x \in K'_m I^0_\mu$ and hence $\pi_1(\lambda) \in K'\pi_1(K'_m J_\mu) \subset K'\pi_1(W)$, which is a contradiction. Using this, as in Lemma 2.5, one can show that there exists an n such that for any $m \geq n$, $\pi_1(\mu) \neq \pi_1(\lambda_1)\cdots\pi_1(\lambda_m)$, with λ_i's commuting with each other and $\pi_1(\lambda_1), \ldots, \pi(\lambda_n) \in K'\pi_1(U) \setminus K'\pi_1(W)$. Now using this and the fact that $\overline{WW'} \subset U$, as in the proof of Lemma 2.6, we get that $\pi_1(\mu)$ has a $K'\pi_1(U)$-decomposition in $M'(G)$. Since G is first countable, as in the proof of Proposition 2.8, there exists an abelian semigroup S_1 such that $\pi_1(\mu)$ has a $K''U$-decomposition for all neighbourhoods U of J', where $K'' = K' \cap S_1$ is a compact group and $J' = \pi_1(J) = \pi_1(J_\mu) \cap S_1$, where $J \subset J_\mu$ is a compact semigroup. Moreover, for any $\pi_1(\alpha) \in S_1$, such that $T_\alpha \subset T_\mu$, $\pi_1(\alpha)$ has a $K''U$-decomposition in S_1.

Now since J is a compact abelian semigroup, it has a maximal idempotent say l, where $l = \omega_L$ for some compact subgroup L of G. Then Jl is a compact abelian group. Let $l' = \pi_1(l)$, $J'' = J'l'$ and let $S_2 = S_1 l'$. Then J'' (resp. S_2) is a compact abelian group (resp. closed abelian semigroup) with identity l'. Let $H = \{x \in I(\mu)| \ xl \in J\}$. Then H is a compact abelian group. Let $h = \omega_H$, $h' = \pi(h)$, $K = K''h'$ and $S_2(h) = S_2h' = S_1h'$. Then K (resp. $S_2(h)$) is a compact abelian group (resp. closed abelian semigroup) with identity h'. Also $J'h' = J''h' = \{h'\}$, $\pi_1(\mu)h' = \pi_1(\mu) \in S_2(h)$. Moreover, as in the proof of

Proposition 2.8, one can get that $\pi_1(\mu)$, and also each $\pi_1(\alpha) \in T_{\pi_1(\mu)}$, has a KU-decomposition for every neighbourhood U of h'. Also, $T_{\pi_1(\mu)}$ is compact. Here, $K = \pi_1(I_\mu^0)h' \cap S_2(h)$. Now as in the Remark before Proposition 4.4 in [S3], we define another equivalence relation '$\cong$' on $S_1(h)$ as:

$$a, b \in S_2(h), \quad a \cong b \quad \text{if } a = kb \text{ for some } k \in K.$$

Let $S' = S_2(h)/\cong$ and let π_2: $S_2(h) \to S'$ be the natural projection. Let $\pi = \pi_2 \circ \pi_1$ and let $S = \pi^{-1}(S')$. Then '$\approx$' is the equivalence relation defined by π. Clearly, S is an abelian semigroup with identity h, $\pi(\mu)$ is bald and infinitesimally divisible in S' and $T_{\pi(\mu)}$ is compact and associatefree (see [S3] for details). Moreover, it is easy to see that any $\alpha \in T_\mu$ satisfies the condition mentioned in the statement of the proposition. ∎

3. Partial homomorphisms

In this section we construct the partial homomorphisms required to show the (shift-)embeddability of the limit of a triangular system.

Given a Hausdorff abelian semigroup S with identity e, for a $\lambda \in S$, a map f_λ: $T_\lambda \to R_+$ is said to be a **λ-norm** if it is continuous at e and it is a partial homomorphism, i.e. $f_\lambda(\lambda_1\lambda_2) = f_\lambda(\lambda_1) + f_\lambda(\lambda_2)$, if $\lambda_1, \lambda_2, \lambda_1\lambda_2 \in T_\lambda$.

LEMMA 3.1: *Let G be an almost periodic group and let H be any compact subgroup of G. Let $\lambda \in M_H^1(G)$ be such that λ is infinitesimally divisible in $M_H^1(G)$. If $\lambda\tilde{\lambda}$ is an idempotent, then $\lambda\tilde{\lambda} = \omega_H$ and hence $\lambda = \delta_x\omega_H = \omega_H\delta_x$ for some $x \in \operatorname{supp}\lambda$.*

Proof: Since G is almost periodic, the finite dimensional irreducible unitary representations separate points of G. For any irreducible unitary representation $(\mathcal{U}, \mathcal{H})$, the map $M^1(G) \to BL(\mathcal{H})$, defined by $\mu \mapsto \int U(g)d\mu$, is a continuous homomorphism.

Let $\lambda\tilde{\lambda} = \omega_{H'}$, for some compact group H' and $H \subset H'$. We have to show that $H' = H$. Let U be a neighbourhood of ω_H in $M_H^1(G)$, such that $\mathcal{U}(U)$ consists of invertible elements in the subalgebra $\mathcal{B}$ generated by $\mathcal{U}(M_H^1(G))$. Since λ is infinitesimally divisible in $M_H^1(G)$, $\mathcal{U}(\lambda)$ is invertible in $\mathcal{B}$. Since $\mathcal{U}(\tilde{\lambda})$ is the adjoint of $\mathcal{U}(\lambda)$, $\mathcal{U}(\lambda\tilde{\lambda}) = \mathcal{U}(\omega_{H'})$ is also invertible in $\mathcal{B}$. That is, there exists $A \in \mathcal{B}$ such that $\mathcal{U}(\omega_{H'})A = A\mathcal{U}(\omega_{H'}) = \mathcal{U}(\omega_H)$ and hence $\mathcal{U}(\omega_{H'}) = \mathcal{U}(\omega_H)\mathcal{U}(\omega_{H'}) = A\mathcal{U}(\omega_{H'})\mathcal{U}(\omega_{H'}) = A\mathcal{U}(\omega_{H'}) = \mathcal{U}(\omega_H)$. Since this holds for all $\mathcal{U}$, we have that $H = H'$ (cf. [He2], Theorems 1.3.3 and 1.3.8), and $\lambda\tilde{\lambda} = \omega_H$. Since $\lambda \in M_H^1(G)$, $\lambda = \omega_H\delta_x = \omega_H\delta_x$. ∎

LEMMA 3.2: *Let G be a totally disconnected locally compact group and let H be a compact subgroup of G. Let $\lambda \in M_H^1(G)$ be infinitesimally divisible in $M_H^1(G)$. If λ is a translate of an idempotent then $\lambda = \omega_H$.*

Proof: Let $\lambda = \delta_x \omega_{H'}$ for some compact subgroup H' of G. Let $\{H_\gamma\}$ be a neighbourhood basis of identity consisting of open compact subgroups of G normalised by H' and let $\{U_\gamma\}$ be the neighbourhood basis of ω_H such that for each n, $\beta(H_\gamma H) > 1/2$ for all $\beta \in U_\gamma$. Now since λ is infinitesimally divisible in S, for any fixed γ, there exist $\lambda_1, \ldots, \lambda_m \in U_\gamma$, commuting with each other, and such that $\lambda = \lambda_1 \cdots \lambda_m$. Since λ_i's commute, $\operatorname{supp} \lambda_i \subset x_i H'$, for some $x_i \in \operatorname{supp} \lambda_i \cap H_\gamma H$, $\operatorname{supp} \lambda_i \subset H_\gamma H'$ for each i. Therefore $\operatorname{supp} \lambda \subset H_\gamma H'$ for all γ.

Also $\bigcap_\gamma H_\gamma H' = H'$, and thus $\operatorname{supp} \lambda \subset H'$. Therefore, $\lambda = \omega_{H'}$, namely λ itself is an idempotent. Clearly, $H \subset H'$. Now since $\lambda = \omega_{H'}$ is also infinitesimally divisible in $M_H^1(H_\gamma H')$, for a fixed γ and $H_\gamma H'$ is a compact group, by Lemma 3.1, $\lambda = \omega_H$. ∎

PROPOSITION 3.3: *Let G be almost periodic. Let S be a closed abelian subsemigroup of $M_H^1(G)$ with identity ω_H. Let $\lambda \in S$ be such that λ is not a translate of an idempotent and λ is infinitesimally divisible in S. Then there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: Since λ is not a translate of an idempotent by Lemma 3.1, $\nu = \lambda\tilde{\lambda}$ is not an idempotent. Hence there exists a continuous finite-dimensional unitary representation $(\mathcal{U}, \mathcal{H})$ of G, such that $\mathcal{U}(\nu)^2 \neq \mathcal{U}(\nu)$ (cf. [He2], Theorems 1.3.3 and 1.3.8). We note that $\mathcal{U}(\omega_H)$ is a (self-adjoint) projection. Let $\mathcal{H}'$ be the range of $\mathcal{U}(\omega_H)$. Let ϱ: $M_H^1(G) \to BL(\mathcal{H}')$ be defined by setting $\varrho(\alpha)$ to be the restriction of $\mathcal{U}(\alpha)$ to $\mathcal{H}'$, for all $\alpha \in M_H^1(G)$. Then ϱ is a continuous homomorphism. Let $\mathcal{H}''$ be the kernel of $\mathcal{U}(\omega_H)$. Then $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. Moreover, $\mathcal{H}''$ is contained in the kernel of $\mathcal{U}(\nu)$, in particular, $\mathcal{U}(\nu)(x) = \mathcal{U}(\nu)^2(x)$ for all $x \in \mathcal{H}''$. Now since $\mathcal{U}(\nu)^2 \neq \mathcal{U}(\nu)$, $\varrho(\nu)^2 \neq \varrho(\nu)$ and since $\varrho(\nu)$ is self-adjoint and positive and $\|\varrho(\nu)\| \leq 1$, this implies that there exists an eigenvalue a of $\varrho(\nu)$ such that $0 < a < 1$. Then $|\det(\varrho(\lambda))|^2 = \det(\varrho(\nu)) < 1$. Since λ is infinitesimally divisible, as shown in Lemma 3.1, $\mathcal{U}(\lambda)$ is invertible in the subalgebra of $BL(\mathcal{H})$ with identity $\mathcal{U}(\omega_H)$. Therefore $\varrho(\lambda)$ is invertible in $BL(\mathcal{H}')$ and hence $|\det(\varrho(\lambda))| > 0$. That is, $0 < |\det(\varrho(\lambda))| < 1$. Hence $\det(\varrho(\alpha)) \neq 0$ for all $\alpha \in T_\lambda$ in S. Now we define f_λ: $T_\lambda \to \mathbb{R}_+$ as follows: $f_\lambda(\alpha) = -\log|\det(\varrho(\alpha))|$, for all $\alpha \in T_\lambda$. Then it is continuous, $f_\lambda(\lambda) > 0$ and if $\lambda_1\lambda_2 \in T_\lambda$, then $f_\lambda(\lambda_1\lambda_2) = f_\lambda(\lambda_1) + f_\lambda(\lambda_2)$. ∎

THEOREM 3.4: *Let G be a compact extension of a closed solvable normal subgroup. Let S be a closed abelian subsemigroup of $M^1_H(G)$ with identity ω_H. Let $\lambda \in S$ be such that the identity $e \in \operatorname{supp} \lambda$ and λ is not an idempotent. Suppose that each $\alpha \in T_\lambda$ is infinitesimally divisible in S and it is supported on $G(\lambda)$. Then there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: Since S is abelian, for any $\alpha \in T_\lambda$, $T_\alpha \subset T_\lambda$, and hence if S' is the closed abelian subsemigroup generated by T_λ in S, then each $\alpha \in T_\lambda$ is infinitesimally divisible in S'. Hence without loss of generality, we may assume that S is generated by T_λ. Then we have that all the measures in S are supported on $G(\lambda)$. Since $e \in \operatorname{supp} \lambda$ and λ is not an idempotent, λ cannot be a translate of an idempotent.

STEP 1: Suppose that G is compact. G is almost periodic and the assertion follows from the previous Proposition. If $G(\lambda)$ is compact, then we may take $G = G(\lambda)$ and the assertion follows.

STEP 2: Now suppose that $G(\lambda)$ is not compact. Let R be a closed normal solvable subgroup such that G/R is compact and let $\pi: G \to G/R$ be the natural projection. Then $\overline{\pi(S)}$ is a compact abelian semigroup with identity $\pi(\omega_H)$ and $\pi(\lambda)$ is infinitesimally divisible in $\pi(S)$.

If $\pi(\lambda)$ is not a translate of an idempotent then there exists a continuous $\pi(\lambda)$-norm on $\pi(S)$ such that $f_{\pi(\lambda)}(\pi(\lambda)) > 0$ (cf. Proposition 3.2). Let $f_\lambda: T_\lambda \to \mathbb{R}_+$ be defined as follows: $f_\lambda(\alpha) = f_{\pi(\lambda)}(\pi(\alpha))$ for all $\alpha \in T_\lambda$. Then f_λ is the desired λ-norm.

STEP 3: Let $\pi(\lambda)$ be a translate of an idempotent. Then since $e \in \operatorname{supp} \lambda$, $\pi(\lambda)$ is an idempotent. $\pi(\lambda)$ is infinitesimally divisible in $\overline{\pi(S)}$, by Lemma 3.1, $\pi(\lambda) = \omega_{\pi(H)}$. Then $G(\lambda) \subset HR$ and since HR/R is compact, without loss of generality we may assume that $G = HR$.

Now we prove the rest of the assertion by induction on the length n of R, i.e. the smallest n such that $R_n = \{e\}$, where $R_1 = \overline{[R, R]}$ and $R_{m+1} = \overline{[R_m, R_m]}$ for all $m \in \mathbb{N}$.

STEP 4: For this step, we only assume that λ is not a translate of an idempotent.

Let $n = 1$. That is, R is abelian. Then $H \cap R$ is normal in G. Let $G' = G/(H \cap R)$ and let $\pi': G \to G'$ be the natural projection. Then $G' = H'R'$, a semidirect product of $H' = H/(H \cap R)$ and $R' = R/(H \cap R)$. The projection of $\pi'(\alpha)$ on $H' = G'/R'$ is $\omega_{H'} = \pi'(\omega_H)$ for all $\alpha \in S$. For any $\alpha \in S$, let α' be the projection of $\pi'(\alpha)$ on R'. Then $\alpha' * \omega_{H'} = \omega_{H'} * \alpha' = \pi'(\alpha)$ (cf. [HS], Lemma 4.3). Let $S' = \overline{\{\alpha' \in M^1(R') \mid \alpha \in S\}}$. Then S' is a closed abelian semigroup with identity δ_e and $\alpha \mapsto \alpha'$ is a continuous homomorphism and λ' is infinitesimally divisible in S'. Suppose λ' is a translate of an idempotent. Since R is abelian, it is almost periodic, by Lemma 3.1, $\lambda' = \delta_x$. This implies that λ is supported on a coset H and it is $\delta_x \omega_H$, a contradiction. Thus λ' is not a translate of an idempotent and from Statement 6.1 of [R2] there exists a continuous λ'-norm $f_{\lambda'}$ on S' such that $f_{\lambda'}(\lambda') > 0$, and hence one can define $f_\lambda(\alpha) = f_{\lambda'}(\alpha')$.

STEP 5: Now suppose that the theorem holds when $n \leq k-1$ and let $n = k$. Let $\pi_1 \colon G \to G/R_1$ be the natural projection. Since R/R_1 is abelian, if $\pi_1(\lambda)$ is not a translate of an idempotent then, from Step 4, there exists a continuous λ'-norm $f_{\pi_1(\lambda)} \colon T_{\pi_1(\lambda)} \to \mathbb{R}_+$ such that $f_{\pi_1(\lambda)}(\pi_1(\lambda)) > 0$. Let $f_\lambda \colon T_\lambda \to \mathbb{R}_+$ be defined as follows: $f_\lambda(\alpha) = f_{\pi_1(\lambda)}(\pi_1(\alpha))$, for all $\alpha \in T_\lambda$, and the assertion follows in this case.

STEP 6: If $\pi_1(\lambda)$ is a translate of an idempotent, then $\pi_1(\lambda)$ is indeed an idempotent say $\omega_{H'}$ for some compact subgroup $H' \subset G/R_1$ and it is infinitesimally divisible in $M^1_{\pi_1(H)}(H')$, by Lemma 3.1, $\pi_1(\lambda) = \omega_{\pi_1(H)}$. Hence $G(\lambda) \subset HR_1$ and we may assume that $G = HR_1$. Since the length of $R_1 = k-1$, the assertion follows by induction. This completes the proof. ∎

COROLLARY 3.5: *Suppose that G and S are as in Theorem 3.4. Let S consist of normal measures. Let $S' = \overline{\{\nu\tilde{\nu} \mid \nu \in S\}}$ (it is an abelian semigroup). Let $\lambda \in S$ be such that λ is not a translate of an idempotent and $\lambda\tilde{\lambda}$ is infinitesimally divisible in S'. Then there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: Let ω_H be the identity in S and S'. For any measure ν, $e \in \operatorname{supp}(\nu\tilde{\nu})$. Hence if $\nu\tilde{\nu} \in T_{\lambda\tilde{\lambda}}$ in S', then $\operatorname{supp}(\nu\tilde{\nu}) \subset G(\lambda\tilde{\lambda})$. Now if $\lambda\tilde{\lambda} = \omega_{H'}$, an idempotent, then it follows from the assumption that $\lambda\tilde{\lambda}$ is infinitesimally divisible in $M^1_H(H')$, which is compact. Hence by Lemma 3.1, $\lambda\tilde{\lambda} = \omega_H$ and hence λ is a translate of ω_H, a contradiction. Therefore, $\lambda\tilde{\lambda}$ is not an idempotent. Now S' and $\lambda\tilde{\lambda}$ satisfy all the conditions in Theorem 3.4 and hence there exists a continuous $\lambda\tilde{\lambda}$-norm $f_{\lambda\tilde{\lambda}}$ on S' such that $f_{\lambda\tilde{\lambda}}(\lambda\tilde{\lambda}) > 0$. If $\lambda_1, \lambda_2, \lambda_1\lambda_2 \in T_\lambda$ in S then $\lambda_1\tilde{\lambda}_1\lambda_2\tilde{\lambda}_2 \in T_{\lambda\tilde{\lambda}}$ in S'. Let $f_\lambda : T_\lambda \to R^+$ be defined as $f_\lambda(\alpha) = f_{\lambda\tilde{\lambda}}(\alpha\tilde{\alpha})$ for all $\alpha \in T_\lambda$. It is continuous and $f_\lambda(\lambda_1\lambda_2) = f_\lambda(\lambda_1)f_\lambda(\lambda_2)$, for any λ_1, λ_2 as above and $f_\lambda(\lambda) = f_{\lambda\tilde{\lambda}}(\lambda\tilde{\lambda}) > 0$. This proves the assertion. ∎

THEOREM 3.6: *Let G be a totally disconnected group which is a compact extension of a closed solvable normal subgroup. Let S be a closed abelian subsemigroup of $M^1_H(G)$ with identity ω_H. Let $\lambda \in S$ be such that λ is not a translate of an* *idempotent and each $\alpha \in T_\lambda$ is infinitesimally divisible in S. Then there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: Since S is abelian for any $\alpha \in T_\lambda$, $T_\alpha \subset T_\lambda$, and hence without loss of generality, we may assume that S is the closed subsemigroup generated by T_λ. Moreover, as in the last part of proof of Proposition 2.8, we can show that $S \subset M^1_H(G(\lambda)) = \omega_H M^1(G(\lambda))\omega_H$.

Suppose that G is compact. Then the assertion follows from Proposition 3.3. If $G(\lambda)$ is compact then, since $S \subset M^1_H(G(\lambda))$, we may assume that $G = G(\lambda)$ and the assertion follows.

Now suppose that $G(\lambda)$ is not compact. Let R be a closed normal solvable subgroup such that G/R is compact and let $\pi : G \to G/R$ be the natural projection. Then $\overline{\pi(S)}$ is a compact abelian semigroup with identity $\pi(\omega_H)$ and $\pi(\lambda)$ is infinitesimally divisible in $\overline{\pi(S)}$.

If $\pi(\lambda)$ is not a translate of an idempotent, then the assertion follows exactly as in Step 2 of Theorem 3.4.

Suppose that $\pi(\lambda)$ is a translate of an idempotent. Then since λ is infinitesimally divisible in $\overline{\pi(S)}$ with identity $\omega_{\pi(H)}$, by Lemma 3.2, $\pi(\lambda) = \omega_{\pi(H)}$. Then $G(\lambda) \subset HR$ and, since HR/R is compact, without loss of generality we may assume that $G = HR$.

Now we prove the rest by induction on the length n of R. Let $n = 1$. Then the assertion follows exactly as in Step 4 of the proof of Theorem 3.4.

Now suppose that the assertion holds when $n \leq k - 1$ and consider the case when $n = k$. Let $\pi_1 : G \to G/R_1$ be the natural projection, where $R_1 = \overline{[R, R]}$ is the commutator subgroup of R. Since R/R_1 is abelian, if $\pi_1(\lambda)$ is not a translate of an idempotent, then the assertion follows as in Step 5 of the proof of Theorem 3.4. Suppose $\pi_1(\lambda)$ is a translate of an idempotent. Then by Lemma 3.2, $\pi_1(\lambda) = \omega_{\pi_1(H)}$ as $\pi_1(\lambda)$ is infinitesimally divisible in $\overline{\pi_1(S)}$. Hence $G(\lambda) \subset HR_1$ and we may assume that $G = HR_1$. Since the length of R_1 is $k - 1$, the assertion follows by induction. This completes the proof. ∎

PROPOSITION 3.7: *Let G be a compact extension of a closed solvable normal subgroup such that G^0 is either compact or nilpotent. Let S be a closed abelian subsemigroup of $M^1(G)$ with identity ω_H. Let $\lambda \in S$ be such that λ is not a translate of an idempotent. Suppose that for any $\alpha \in T_\lambda$ and any neighbourhood U of ω_H in S, there exist $\lambda_1, \ldots, \lambda_m \in U$ and $u \in S$ such that $\lambda = \lambda_1 \cdots \lambda_m u$, where $u = \delta_x * \omega_H$ for some $x \in G^0$. Then there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: As before, we may assume that S is generated by T_λ. Suppose λ is not a translate of an idempotent. Now suppose $\pi: G \to G/G^0$ is the natural projection. Then $\pi(\alpha)$ is infinitesimally divisible in $\overline{\pi(S)}$ for all $\alpha \in T_\lambda$. If $\pi(\lambda)$ is not a translate of an idempotent, then the assertion follows from Theorem 3.6. Let $\pi(\lambda)$ be a translate of an idempotent. Then since $\pi(\lambda)$ is infinitesimally divisible in $\overline{\pi(S)}$, by Lemma 3.2, $\pi(\lambda) = \omega_{\pi(H)}$. Also, for any $\alpha \in T_\lambda$, $\pi(\alpha) = \omega_{\pi(H)}$. Hence we may assume that $G = HG^0$. First suppose that G^0 is compact. Then $G = HG^0$ is compact and the assertion follows from Proposition 3.3.

Now suppose G^0 is nilpotent. Since $H \cap G^0$ is a compact subgroup of G^0, $H \cap G^0$ is central in G^0 and hence normal in G. Then $G_1 = G/(H \cap G^0) = H' \cdot G'$, a semidirect product of $H' = H/(H \cap G^0)$ and $G' = G^0/(H \cap G^0)$. If $\alpha \in S$, then the image of α on G_1/G' is $\omega_{H'}$. Hence as in Step 4 of the proof of Theorem 3.4, we can define S' consisting of α' where α' is a projection of α on G', for all $\alpha \in S$. Then S' is a closed abelian semigroup with identity δ_e. Let K be the maximal compact (central) subgroup of G^0 and let π': $G' \to G^0/K$ be the natural projection. Then G^0/K is a simply connected nilpotent Lie group and, if $\pi'(\lambda')$ is not a translate of an idempotent, by Theorem 5.1 of [S3], there exists a continuous $\pi'(\lambda')$-norm on $\overline{\pi'(S')}$ and one can define a λ-norm correspondingly. Let $\pi'(\lambda')$ be a translate of an idempotent, then since G^0/K is simply connected, $\pi'(\lambda'\tilde{\lambda}') = \delta_e$ and hence $\operatorname{supp}(\lambda'\tilde{\lambda}') \subset K' = K/(H \cap G^0)$, which is compact and central in G'. So for any $\alpha' \in T_{\lambda'}$, $\alpha'\tilde{\alpha}' = \tilde{\alpha}'\alpha' \in T_{\lambda'\tilde{\lambda}'}$, $\alpha'\tilde{\alpha}'$ is supported on K' and it is infinitesimally divisible in $M^1(K')$. Now if $\lambda'\tilde{\lambda}'$ is not an idempotent on K', the assertion follows from Proposition 3.3. If $\lambda'\tilde{\lambda}'$ is an idempotent then, by Lemma 3.1, $\lambda'\tilde{\lambda}' = \delta_e$ in $M^1(K')$. This implies that $\operatorname{supp} \lambda\tilde{\lambda} \subset H$ and hence $\lambda\tilde{\lambda} = \omega_H$. In particular, λ is a translate of an idempotent which is a contradiction. This completes the proof. ∎

PROPOSITION 3.8: *Let G be any locally compact group and let H be any open subgroup of G. Let S be any closed abelian subsemigroup of $M^1_H(G)$ such that S consists of normal measures. If $\lambda \in S$ is not a translate of an idempotent, then there exists a λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: Let $\mathcal{B}$ be the Banach algebra generated by S and $\tilde{S}$ in $M(G)$, where $\tilde{S} = \{\tilde{\nu} | \ \nu \in S\}$. Then $\mathcal{B}$ is a commutative C^*-algebra with identity ω_H. Since H is open, if $\{\lambda_n\} \subset S$ is such that $\lambda_n \to \omega_H$ weakly then $\lambda_n \to \omega_H$ in the norm topology.

Suppose that λ is not a translate of an idempotent and, if possible, suppose that $\lambda\tilde{\lambda} \in \mathcal{B}$ is an idempotent, say $\omega_{H'}$, for some compact group H'. Then $\operatorname{supp} \lambda \subset H'x = xH'$ for any $x \in \operatorname{supp} \lambda$. The spectrum of $\lambda\tilde{\lambda}$ on $\mathcal{B}$ is contained in $\{0,1\}$ and hence, for any continuous complex homomorphism f on $\mathcal{B}$, $f(\lambda(\lambda\tilde{\lambda})) = f(\lambda)f(\lambda\tilde{\lambda}) = f(\lambda)$ and hence, by the Gelfand–Naimark Theorem (cf. [Ru], Theorems 11.9, 11.18), $\lambda(\lambda\tilde{\lambda}) = \lambda\omega_{H'} = \lambda$ and therefore $\lambda = \omega_{H'}x$ is a translate of an idempotent, a contradiction. Therefore $\lambda\tilde{\lambda}$ is not an idempotent. Then there exists a continuous complex homomorphism f: $\mathcal{B} \to \mathbf{C}$ such that $0 < f(\lambda\tilde{\lambda}) < 1$ (cf. [Ru], Theorems 11.9, 11.18). Now we define f_λ: $T_\lambda \to \mathbb{R}_+$ as follows: $f_\lambda(\alpha) = -\log f(\alpha\tilde{\alpha})$, for all $\alpha \in T_\lambda$. Clearly f_λ is a λ-norm such that $f_\lambda(\lambda) > 0$. ∎

THEOREM 3.9: *Let G be a discrete group and let S be a closed abelian subsemigroup of $M^1(G)$ with identity ω_H. Let $\lambda \in S$ be such that λ is not a translate of an idempotent and each $\alpha \in T_\lambda$ is infinitesimally divisible in S. Then there exists a λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$ if any one of the following conditions is satisfied:*

(1) *G is a finite extension of a solvable (normal) subgroup,*
(2) *S consists of normal measures, or*
(3) *G is a linear group over a locally compact field of characteristic zero.*

Proof: If (1) is satisfied, the assertion is obvious from Theorem 3.6. If S consist of normal measures, then the assertion follows from Proposition 3.8.

Now let G be a discrete linear group over a locally compact field of characteristic zero. As in the proof of Theorem 3.6, $\operatorname{supp}\alpha \subset G(\lambda)$ for all $\alpha \in T_\lambda$ and hence, without loss of generality, we may assume that S is generated by T_λ and that $G = G(\lambda)$. If G is amenable then, by Tits' theorem (cf. [T], Theorem 1), G is a finite extension of a solvable normal subgroup and hence the assertion follows from (1). Now let G be nonamenable. Let $L^2(G)$ be the set of all square integrable functions on G with respect to the left invariant Haar measure on G; let $BL(L^2(G))$ be the Banach algebra of all bounded linear operators on $L^2(G)$. Let ψ: $M(G) \to BL(L^2(G))$ be the canonical representation. It is faithful and continuous with respect to the norm topology on both the spaces and $\|\psi(\lambda)\| \leq \|\lambda\|$ (cf. [HR], Theorem 20.11). Moreover, since G is discrete, as before, $\{\lambda_n\} \subset S$ converges to ω_H weakly if and only if it converges in the norm topology and hence $\{\psi(\lambda_n)\}$ converges in the norm topology. Let $P = \psi(\omega_H)$ and let $\mathcal{B}'$ be the commutative Banach algebra generated by $\psi(S)$, with identity P. Since λ is infinitesimally divisible in S, so is $\psi(\lambda)$ in $\mathcal{B}'$ and hence it is invertible in $\mathcal{B}'$.

Since $G = G(\lambda)$ is nonamenable, the spectral radius of $\psi(\lambda)$ on $BL(L^2(G))$ is less than 1 (cf. [DG]). Let $\mathcal{C}' = PBL(L^2(G))P$. Then $\mathcal{C}'$ is a Banach algebra with identity P and $\psi(\lambda) \in \mathcal{C}'$. Now if for some $c \in \mathbf{C}$, if $\psi(\lambda) - cI$ is invertible in $BL(L^2(G))$ with inverse A, then it is easy to see that $\psi(\lambda) - cP$ is invertible in $\mathcal{C}'$ with inverse PAP. Therefore the spectral radius of $\psi(\lambda)$ on $\mathcal{C}'$ is also less than 1. Since $\psi(\lambda)$ is invertible in $\mathcal{B}' \subset \mathcal{C}'$, the specral radius of $\psi(\lambda)$ is positive on $\mathcal{C}'$. That is, there exists an a in the spectrum of $\psi(\lambda)$ on $\mathcal{C}'$ and hence on $\mathcal{B}'$ such that $0 < |a| < 1$. Now since $\mathcal{B}'$ is commutative, by the Gelfand–Naimark Theorem there exists a continuous complex-valued function $f : \mathcal{B}' \to \mathbf{C}$ such that $f(\lambda) = a$. Let $f_\lambda : T_\lambda \to \mathbb{R}_+$ be defined as follows: $f_\lambda(\alpha) = -\log|f(\alpha)|$, for all $\alpha \in T_\lambda$. Clearly f_λ satisfies desired conditions. ∎

PROPOSITION 3.10: *Let G be a closed subgroup of $GL(n, \mathbb{Q}_p)$. Let S be a closed abelian subsemigroup of $M^1(G)$ with identity ω_H such that H is open in G. Let $\lambda \in S$ be such that λ is not a translate of an idempotent and each $\alpha \in T_\lambda$ is infinitesimally divisible in S. Then there exists a λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof: As in the proof of Theorem 3.6, without loss of generality we may assume that S is generated by T_λ and $G = G(\lambda)$. Let G be nonamenable, let $\psi : G \to BL(L^2(G))$ be the canonical representation and let $\mathcal{B}'$ be the commutative Banach algebra generated by $\psi(S)$ in $BL(L^2(G))$. Then $\psi(\omega_H)$ is the identity in $\mathcal{B}'$. Since $G = G(\lambda)$ is nonamenable, as in the last part of the proof of Theorem 3.9, the spectral radius of $\psi(\lambda)$ on $\mathcal{C}' = \psi(\omega_H)BL(L^2(G))\psi(\omega_H)$ is less than 1. Since H is open, we have that if $\{\lambda_n\} \subset S$ converges to ω_H weakly then it converges in the norm topology, and hence $\{\psi(\lambda_n)\}$ converges to $\psi(\omega_H)$ in the norm topology. Also, since λ is infinitesimally divisible is S, so is $\psi(\lambda)$ in $\mathcal{B}'$ and hence it is invertible in $\mathcal{B}' \subset \mathcal{C}'$. Therefore, the spectral radius of $\psi(\lambda)$ on $\mathcal{C}'$ is positive and hence there exists a λ-norm as in the last part of the proof of Theorem 3.9.

Now suppose that G is amenable. Let $\tilde{G}$ be the Zariski closure of G in $GL(n, \mathbb{Q}_p)$. Let R be the radical of $\tilde{G}$, i.e. R is the maximal Zariski connected solvable normal subgroup of $\tilde{G}$. Then $\tilde{G}/R$ is semisimple. Let π: $\tilde{G} \to \tilde{G}/R$ be the natural projection. Then $\overline{\pi(G)}$ is Zariski dense in $\tilde{G}/R$. Also, $\overline{\pi(G)}$ is amenable (cf. [Z], Lemma 4.1.13). Therefore, since $\tilde{G}/R$ is semisimple, using Furstenberg's Lemma (Lemma 3.2.1 in [Z] or Lemma 1 in [Sh]), one can show as in the Corollary of [Sh] that $\overline{\pi(G)}$ is compact. That is, $\overline{GR}$ is a compact extension of a closed solvable normal subgroup. Now the assertion follows from Theorem 3.6. ∎

4. Embeddability

In this section we find the embedding of the limit using the results on infinitesimal divisibility and existence of λ-norm.

Let us first recall Theorem 2.3 of [S3] as it will be very useful here.

THEOREM 4.1: *Let S be an abelian Hausdorff semigroup with identity e. Let a be the limit of an infinitesimal triangular system in S. Suppose that a is bald in S and that T_a is compact and associatefree. Suppose for all $s \in T_a \setminus \{e\}$, there exists an s-norm f_s such that $f_s(s) > 0$. Then there exists a continuous homomorphism $\phi : R_+ \to S$ such that $\phi(1) = a$.*

Proof of Theorem 1.1: As in Proposition 2.8, there exists an abelian semigroup $S \subset M_H^1(G(\mu))$ with identity ω_H such that μ is bald, infinitesimally divisible and T_μ in S is compact and associatefree. Let $\lambda \in T_\mu \setminus \{\omega_H\}$. Then, as in Proposition 2.8, λ is infinitesimally divisible in S. If G is totally disconnected, then by Lemma 3.2, λ is not a translate of an idempotent. If λ is symmetric and it is a translate of an idempotent (say), $\omega_{H'}$, then $\omega_{H'} \in T_\mu$ and, since μ is bald, $H = H'$ and hence $\lambda^2 = \omega_H$, the identity in S. Since T_μ is associatefree, it is a contradiction. Since $S \subset M^1(G(\mu))$ and $G(\mu) \subset L$, without loss of generality we may assume $G = L$ (where L is as in the hypothesis). Now from Theorems 3.6, Corollary 3.5 and Proposition 3.3 there exists a λ-norm such that $f_\lambda(\lambda) > 0$. Hence the assertion follows from Theorem 4.1.

Proof of Theorem 1.2: Let G be any discrete group. As in Proposition 2.8, there exists an abelian semigroup $S \subset M_H^1(G(\mu))$ with identity ω_H such that μ is bald and infinitesimally divisible in S and T_μ is compact and associatefree. Moreover, S consists of normal measures if Δ is normal. Let $\lambda \in T_\mu \setminus \{\omega_H\}$; then, as in the proof of Theorem 1.1, λ is not a translate of an idempotent. Now if $G(\mu)$ is a finite extension of a solvable group, then the assertion follows from Theorem 1.1. If $G(\mu)$ is a discrete linear group over a locally compact field of characteristic zero or if Δ is normal, then, by Theorem 3.9, there exists a λ-norm f_λ such that $f_\lambda(\lambda) > 0$. Now the assertion follows from Theorem 4.1. ∎

THEOREM 4.2: *Let G be a locally compact first countable group such that G is a compact extension of a closed solvable normal subgroup. Let Δ be a commutative infinitesimal triangular system converging to μ. Suppose that $I_\mu^0/(I_\mu^0 \cap Z)$ is compact. If*

(1) G^0 is compact or nilpotent, or if

(2) Δ is normal,

then μ is weakly infinitely divisible; moreover, if I_μ^0/Z^0 is compact then $x\mu$ is embeddable for some $x \in I_\mu^0$; in particular, if $I_\mu^0 = Z^0$ then μ is infinitely divisible, and it is embeddable if Z^0 is arcwise connected.

Remarks: (1) In the above theorem, if the group G is Lie projective (instead of first countable) and if $I_\mu^0 = Z^0$ (and all the other conditions are satisfied) then, by Proposition A.2 (see Appendix), μ is infinitely divisible in a compact subset of $M^1(G)$ and hence $x\mu$ is embeddable for some $x \in Z^0$; μ itself is embeddable if Z^0 is arcwise connected.

(2) If G is connected and nilpotent, then the condition that $I_\mu^0 = Z^0$ is the same as the condition that I_μ^0/Z is compact.

(3) If G is totally disconnected then $I_\mu^0 = Z^0 = \{e\}$ and hence some parts of Theorems 1.1 and 1.2 follow trivially from Theorem 4.2. ∎

Proof of Theorem 4.2: Let G, Δ, μ be as given. Since $I_\mu^0/(I_\mu^0 \cap Z)$ is compact, by Proposition 2.9, there exist an abelian semigroup $S \subset M_H^1(G)$ and an equivalence relation $\approx$ and the corresponding projection $\pi : S \to S' = S/\approx$ such that $\pi(\mu)$ is bald and infinitesimally divisible in S' and $T_{\pi(\mu)}$ is compact and associatefree. Moreover, any $\alpha \in T_\lambda$ satisfies the condition mentioned in the proposition. Here $\pi = \pi_1\pi_2$, where π_1 and π_2 are as in the proof of Proposition 2.9. Since $\ker \pi_2$ is compact, it is easy to see that $T_{\pi_1(\mu)}$ (in $\pi_1(S)$) is compact and $\pi_1(x\mu)$ is infinitesimally divisible in $\pi_1(S)$ for some $x \in I_\mu^0$. Also, for any $\alpha \in T_\mu$ (in S), given any neighbourhood U of ω_H in S, there exist $\alpha_1, \ldots, \alpha_n \in U$ and $a \in I_\mu^0$ such that $\delta_a * \omega_H \in S$ and $\alpha = a\alpha_1 \cdots \alpha_n$.

Let G be a compact extension of a closed solvable normal subgroup and let G^0 be compact or nilpotent or let Δ be normal. Let $\lambda \in T_\mu$ be such that $\pi(\lambda) \in T_{\pi(\mu)} \setminus \{\pi(\omega_H)\}$. If possible, suppose $\lambda = \omega_{H'} * \delta_x$. If Δ is normal, then the elements of S are normal and the above condition implies that $\omega_{H'} = \lambda\tilde{\lambda}$ is infinitesimally divisible in $M_H^1(H')$ and hence, by Lemma 3.1, $\lambda\tilde{\lambda} = \omega_H$ and $\lambda = \omega_H * \delta_x$.

Let π': $G \to G/G^0$ be the natural projection. Then $H \subset H'$ and, from the condition above, we get that $\pi'(\lambda)$ is infinitesimally divisible in $M_{\pi'(H)}^1(G/G^0)$ and hence, by Lemma 3.2, $\pi'(\lambda) = \pi'(\omega_H) = \pi'(\alpha)$ for all $\alpha \in T_\lambda$ (in S). That is, $x \in G^0$, $H' \subset HG^0$ and $\lambda\tilde{\lambda}$ is infinitesimally divisible in $M_H^1(HG^0)$. Now if G^0 is compact, then so is HG^0 and hence, by Lemma 3.1, $\lambda\tilde{\lambda} = \omega_H$ and $\lambda = \omega_H * \delta_x$. If G^0 is nilpotent, then $H \cap G^0$ is central in G^0 and normal in HG^0. Let π'': $HG^0 \to HG^0/(H \cap G^0)$ be the natural projection. Here, $HG^0/(H \cap G^0)$ is a semidirect product of $H_1 = H/(H \cap G^0)$ and $G_1 = G^0/(H \cap G^0)$. As in the proof of Proposition 3.7, for $\alpha \in M_H^1(HG^0)$, let α' be the projection of $\pi''(\alpha)$ on G_1. Then $\pi''(\alpha) = \pi''(\omega_H) * \alpha' = \alpha' * \pi''(\omega_H)$. Moreover, for any $\alpha \in T_\lambda$, $\operatorname{supp} \alpha \subset H'y \subset HG^0$, where $y \in G^0$ and hence $\operatorname{supp} \alpha' \subset \pi''((H' \cap G^0)y)$, where $\pi''(H' \cap G^0)$ is central in G_1 and hence α' is normal. Therefore, $\lambda'\tilde{\lambda}'$ is infinitesimally divisible in $M^1(\pi''(H'\cap G^0))$ and hence, by Lemma 3.1, $\lambda'\tilde{\lambda}' = \delta_e$. Therefore $\lambda = \omega_H * \delta_x$.

Thus in all the three cases, $\lambda = \omega_H * \delta_x$ for some $x \in I_\mu \cap N(H)$, where $N(H)$ is the normaliser of H and, for any $\alpha \in T_\lambda$, $\alpha = \omega_H * \delta_y$ for some $y \in I_\mu \cap N(H)$. Now from the above condition, the image of x on $L = I_\mu \cap N(H)/(I_\mu^0 H \cap N(H))$ is infinitesimally divisible and hence it is trivial as L is totally disconnected. This implies that $x \in I_\mu^0 H \cap N(H)$ and hence $\pi(\lambda) = \pi(\omega_H)$, a contradiction. Hence λ is not a translate of an idempotent and, by Corollary 3.5 and Proposition 3.7, there exists a λ-norm on S, f_λ: $T_\lambda \to \mathbb{R}_+$ such that $f_\lambda(\lambda) > 0$. Clearly, $f_\lambda(\omega_H) = 0$ and , if $u \in T_\lambda$ is invertible, then $u^{-1} \in T_\lambda$ and $f_\lambda(u) + f_\lambda(u^{-1}) = f_\lambda(\omega_H) = 0$, and hence $f_\lambda(u) = 0$. Now if $\nu \approx \nu'$ for some $\nu, \nu' \in T_\lambda$ in S then $\nu = u\nu'$, for some $u \in T_\lambda$ which is invertible, and hence $f_\lambda(\nu) = f_\lambda(\nu')$. Therefore, one can define a $\pi(\lambda)$-norm $f_{\pi(\lambda)}$ on S' in a standard manner such that $f_{\pi(\lambda)}(\pi(\lambda)) = f_\lambda(\lambda) > 0$. Now from Theorem 4.1, $\pi(\mu)$ can be embedded in a continuous one-parameter semigroup. That is, for each n, there exist $x_n \in I_\mu^0$ and $\lambda_n \in T_\mu$ (in S), such that $\mu = x_n \lambda_n^n$. That is, μ is weakly infinitely divisible.

Now let I_μ^0/Z^0 be compact. Then we can take the relation $\sim$ to be as follows: $\lambda \sim \nu$ if $\lambda = z\nu$ for some $z \in Z^0$ and accordingly define $\pi_1 : M^1(G) \to M^1(G)/\sim$ in the proof of Proposition 2.9. Then we have a semigroup S and S' accordingly as constructed in Proposition 2.9 and clearly, for T_μ in S, T_μ/Z^0 is relatively compact. Let $\mu_n = y_n \lambda_{n!}^{n!}$ for some sequences $\{y_n\}$ in $I_\mu^0 \cap S$ and $\{\lambda_{n!}\}$ in T_μ (in S). Then $y_n = k_n z_n$ for some relatively compact sequence $\{k_n\}$ in I_μ^0 and a sequence $\{z_n\} \in Z^0$. Since Z^0 is a connected abelian group each z_n is divisible, and hence $k_n^{-1}\mu = ((z_n/n!)\lambda_{n!})^{n!}$. Let y be a limit of $\{k_n^{-1}\}$. Since Z^0 is compactly generated, by Lemma 3.2 of [S2], $y\mu$ is infinitely divisible in a compact subset of S and hence $x\mu$ is embeddable for some $x \in I_\mu^0$. Now if $I_\mu^0 = Z^0$, then $x \in Z^0$ is infinitely divisible and hence μ is infinitely divisible in S. If Z^0 is arcwise connected, then x^{-1} is embeddable in a continuous one-parameter semigroup $\{x_t\} \subset Z^0$. Hence μ is embeddable. ∎

Proof of Theorem 1.5: Let G be any locally compact group and let $\{\nu_i\}$ be a sequence converging to δ_e such that $\nu_i^{k_i} \to \mu$ for some unbounded sequence $\{k_i\}$. Suppose that $I_\mu^0/(I_\mu^0 \cap Z)$ is compact. Then as in the proof of Proposition 2.9, we define an equivalence relation '$\sim$' and π_1: $M^1(G) \to M'(G) = M^1(G)/\sim$ and get that $\pi(\mu)$ has a $K'\pi_1(U)$-decomposition for any small neighbourhood U of J_μ such that $T_{\pi_1(\mu)} \cap K'\overline{\pi_1(U)}$ is compact, where $K' = \pi_1(I_\mu^0)$. From the construction of the decomposition, it is easy to see that $\pi_1(\mu) = \pi_1(\lambda^m)$ for some λ and m, where $\pi_1(\lambda) \in K'\pi_1(U)$ is a limit of some sequence $\{\pi_1(\nu_i^{l_i})\}$; here we say that $\pi_1(\mu)$ has an *equal decomposition* in $K'\pi_1(U)$ for every neighbourhood U of J_μ. Clearly, each $\pi_1(\lambda)$ also has an equal decomposition in $K'\pi_1(U)$ for every neighbourhood U of J_μ. Let $\{U_\alpha\}$ be a neighbourhood basis of J_μ such that $U_\alpha \subset U$ and $T_{\pi_1(\mu)} \cap \overline{K'\pi_1(U)}$ is relatively compact. Let $\pi_1(\mu) = \pi_1(\lambda_\alpha^{m_\alpha})$ for some m_α and λ_α such that $\pi_1(\lambda_\alpha) \in K'\pi_1(U_\alpha)$.

If $\mu = \delta_x * \omega_H$, for some $x \in I_\mu^0$, then the assertion follows easily. Now suppose that $\mu \neq \delta_x * \omega_H$ for any $x \in I_\mu^0$. Then we have that for every n there exist λ_n and $k_n > n$ such that $\pi_1(\lambda_n) \in K'\pi_1(U)$ and $\pi_1(\mu) = \pi_1(\lambda_n^{k_n})$. If possible, suppose for fixed n, $\pi_1(\mu) = \pi_1(\lambda_\alpha^{m_\alpha})$ for $\lambda_\alpha \in K'\pi_1(U_\alpha)$ and $m_\alpha \leq n$ for all α. Then $\pi_1(\mu) \in (K'\pi_1(U_\alpha))^n$ for all α and hence $\pi_1(\mu) \in K'\pi_1(J_\mu)$ and $\mu = \delta_x * \omega_H$ for some $x \in I_\mu^0$, a contradiction.

Now we have that for every n, $\pi_1(\mu) = \pi_1(\lambda_n^{k_n})$, where $k_n > n$ and $\pi_1(\lambda_n) \in K'\pi_1(U)$. Hence one can choose $\{\pi_1(\lambda_n)\}$ to be a convergent sequence with limit $\pi_1(\lambda)$. Since each $\pi_1(\lambda_n)$ and hence $\pi_1(\lambda_n)^m$ has an equal decomposition in $K'\pi_1(U)$ for every neighbourhood U of J_μ, as in the proof of Proposition 2.8, one can show that $A = \{\pi_1(\lambda_n^m) \mid m \leq k_n\} \subset T_{\pi_1(\mu)}$ is relatively compact. Since $\pi_1(\lambda^n) \in A$ for all n, we have that $\pi_1(\lambda)$ generates a compact semigroup, (say) G_λ, and hence for some sequence $\{l_n\}$, $\{\pi_1(\lambda)^{l_n}\}$ converges to an idempotent, (say) $h \in T_{\pi_1(\mu)}$, $h = \pi_1(\omega_H) \in \pi_1(J_\mu)$, where $H \subset I(\mu)$. This implies that $\pi_1(\lambda)h = \pi_1(\delta_x\omega_H)$ for some $x \in I_\mu$. Since each λ_n and hence λ has an equal decomposition in $K'\pi_1(U)$ for every neighbourhood U of J_μ and $h \in J_\mu$, we get that $x \in I_\mu^0$, i.e. $\pi_1(\delta_x) \in K'$ and hence $\pi_1(\lambda) \in K'\pi_1(J_\mu)$. Now for any n, let $\pi_1(\nu_n)$ be a limit of $\{\pi_1(\lambda_n)^{[k_n/n]}\}$ for large n. Then $\pi_1(\mu) = \pi_1(\nu_n^n)\gamma_n = \pi_1(\nu_n^n)h\gamma_n = \pi_1(\nu_n^n a_n)$, where $\gamma_n \in G_\lambda$, the closed semigroup generated by λ in $K'\pi_1(J_\mu)$, and $a_n \in I_\mu^0$. (See the proof of Theorem 3.6 in [S2] for details.) Hence for each n, $\mu = \nu_n^n x_n$ for some $x_n \in I_\mu^0$, that is, μ is weakly infinitely divisible. Now the proof of the rest of the Theorem is same as the last part of the proof of Theorem 4.2. So we will not repeat it here. ∎

Proof of Theorem 1.6: Let Δ be a commutative ω_K-infinitesimal triangular system converging to μ for some compact open subgroup K of G. Let G be totally disconnected or let Δ be symmetric; then it is easy to see that the proof of Proposition 2.8 holds for such a Δ and there exists a closed commutative subsemigroup S in $M^1(G)$ with identity $\omega_{K'}$ such that $K \subset K'$, $\mu \in S$, μ is bald and infinitesimally divisible in S and T_μ (in S) is compact and associatefree. Now since K is open, so is K'. Moreover, S consists of normal measures if Δ is normal. Let $\lambda \in T_\mu \setminus \{\omega_{K'}\}$. Then as in the proof of Theorem 1.1, λ is not a translate of an idempotent. Now if G is a closed subgroup of $GL(n, \mathbb{Q}_p)$ (resp. if

Δ is normal) then, by Proposition 3.10 (resp. by Proposition 3.8), there exists a λ-norm f_λ such that $f_\lambda(\lambda) > 0$ and the assertion follows from Theorem 4.1. ∎

Appendix

Here we present some useful results on structure of Lie groups and measures.

A.1 PROPOSITION: *Let G be a Lie group with finitely many connected components. Then G can be embedded as a closed subgroup in a Lie group H with finitely many connected components such that, if R' is the solvable radical of H, then the center of H^0/R' is finite.*

Proof: If R is the radical of G and the center of G^0/R is finite, then there is nothing to prove. Now let the center of G^0/R be infinite. By Levi decomposition there exists a semisimple subgroup S of G^0 such that $G^0 = S.R$, a semidirect product, and the center of S is infinite (discrete). Let $C = Z \cap S$, where Z is the center of G^0. Then C is a discrete central subgroup in G^0 and it is normal in G. Now consider the adjoint representation Ad_{G^0} of G^0 over its Lie algebra. Then $\mathrm{Ad}_{G^0} S$ is a connected semisimple matrix group and hence its center is finite. Since S/C is isomorphic to $\mathrm{Ad}_{G^0} S$, the center of S/C is also finite. There exists a subgroup A of C such that C/A is finite and A is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Let $D = \bigcap_{x \in G} xAx^{-1}$. Since C is central in G^0 and normal in G, D is an intersection of finitely many conjugates of A, each of which is a subgroup of finite index in C, and hence D is a subgroup of finite index in A and it is isomorphic to $\mathbb{Z}^n$. Clearly D is normal in G. One can extend the action of G on $D = \mathbb{Z}^n$ to the action on $\mathbb{R}^n$. Using this action we construct $G_1 = G \cdot \mathbb{R}^n$. Let $D' = \{(d, d) |\ d \in D\}$. Then D' is a discrete central subgroup of G_1^0. It is also normal in G_1. Let $H = G_1/D'$. Now a straightforward calculation shows that $R' = (DR \cdot \mathbb{R}^n)/D'$ is the radical of H and the center of H^0/R' is finite. ∎

A.2 PROPOSITION: *Let G be a locally compact group which is the projective limit of a projective system $(G_\alpha, p_{\alpha\beta}, \mathbf{A})$ of the Lie groups $G_\alpha = G/K_\alpha$ $(\alpha \in \mathbf{A})$, where $(K_\alpha)_{\alpha \in \mathbf{A}}$ is a descending system of compact normal subgroups of G satisfying $\bigcap_{\alpha \in \mathbf{A}} K_\alpha = \{e\}$. Let $\mu \in M^1(G)$. If, for every $\alpha \in \mathbf{A}$, $p_\alpha(\mu)$ is infinitely divisible in a compact set A_α such that $p_{\alpha\beta}(A_\beta) \subset A_\alpha$, then μ is infinitely divisible in a compact subset of $M^1(G)$. In particular, μ is infinitely divisible if $p_\alpha(\mu)$ is infinitely divisible in B_α, where $B_\alpha = \{\lambda \in M^1(G_\alpha) |\ \lambda(\pi_\alpha(F)) \geq \delta\}$ for a fixed compact set $F \subset G$ and a fixed $\delta > 0$.*

The proof of the first part is same as Theorem 6.6.1 in [He2]. The second part follows from Lemma 2.1 as $A_\alpha = T_{p_\alpha(\mu)} \cap B_\alpha$ is compact and $p_{\alpha\beta}(A_\beta) \subset A_\alpha$.

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